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What Makes Pedagogical Content Knowledge "Pedagogical"? Reconnecting PCK to Its Deweyan Foundations

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In this theoretical paper, I review the history of research in educational psychology that inspired Shulman's notion of pedagogical content knowledge (PCK) and I critically examine interpretations of PCK reflected in prominent theoretical frameworks for mathematical knowledge for teaching (MKT). I propose a theory of PCK—grounded in radical constructivism and Piaget's genetic epistemology—that addresses limitations of these prominent frameworks. I conclude with a description of what makes PCK in the proposed theory "pedagogical" and describe a research agenda that reconnects MKT scholarship to its Deweyan philosophical foundations.

> *"For about half a century behaviorists have worked hard to do away with 'mentalistic' notions ... It is up to future historians to assess just how much damage this mindless fashion has wrought ... Since behaviorism is by no means extinct, damage continues to be done" (von Glasersfeld, 2007, p. 13).*

In recent decades the number of empirical and theoretical reports on issues related to the nature of mathematics teachers' knowledge have increased substantially. These contributions regularly acknowledge the pioneering work of Lee Shulman (1986, 1987), who is credited for initiating this thriving area of research. Such acknowledgement is often more than a deferential nod—most of these studies establish their relevance by proposing to extend or refine Shulman's conceptualization of teachers' knowledge base, especially his influential notion of pedagogical content knowledge (PCK; 1986). In this theoretical paper, I argue that the field's commitment to elaborating Shulman's framework, fruitful though it has been, has led

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current research on mathematics teachers' knowledge down a cul-de-sac where certain epistemological assumptions about PCK have contributed to a body of literature reflecting features of the process-product research paradigm (Dunkin & Biddle, 1974)—and the behaviorism that influenced it—of which Shulman's work was a timely and influential critique.

I begin by providing an abbreviated history of research in educational psychology that inspired Shulman's seminal contributions. I then review Shulman's (1986, 1987) conceptualization of teachers' knowledge base and critically examine interpretations of PCK foundational to prominent theoretical frameworks for mathematical knowledge for teaching (MKT). Thereafter, I propose a theory of pedagogical content knowledge—grounded in radical constructivism (von Glasersfeld, 1995), Piaget's genetic epistemology (Piaget, 1971), and empirical research (Tallman 2015, 2021; Tallman & Frank, 2020)—that addresses limitations of these prominent frameworks. I conclude with a description of what makes PCK in the proposed theory "pedagogical" and describe a research agenda that reconnects MKT scholarship to its Deweyan philosophical foundations.

Research on Teachers' Knowledge

Research on teacher knowledge has its origins in John Dewey's psychology of school subjects. Dewey's psychology was based on his commitment to elevate the experience of the learner to the status traditionally afforded in educational research to the canonical subject matter of the curriculum. Dewey reified this fundamental principle with the phrase, "psychologizing the subject matter" (Dewey, 1902). This now familiar expression encapsulated Dewey's proposal for educational psychologists to both explicate the experiential basis of the facts, concepts, and ways of reasoning that comprise the subject matter of a discipline, and to identify the capacities and proficiencies that these facts, concepts, and ways of reasoning enable (Dewey, 1902). To Dewey, psychologizing the subject matter meant to explore the experiential reality of learners as they engage with academic subjects, and to characterize this reality in psychological terms. The psychologized subject matter thus maintains two forms of fidelity: (1) to the current and potential experience of the learner and (2) to the content of an academic discipline, including the psychological processes of its past and present expert practitioners (Shulman & Quinlan, 1996, p. 402).

Dewey's vision for a psychology of school subjects that acknowledges the interdependence between the subject matter of the curriculum and the psychological processes from which it originated—as well as those engaged in by learners—dissipated during the period in educational psychology from the 1930s through the 1950s. This period privileged the development of general learning theories uninformed by the nature of the experiences out of which the substantive and syntactic structures[1] of particular academic disciplines emerged (Shulman & Quinlan, 1996, p. 400). Edward Lee Thorndike was most influential to the dissolution of Dewey's psychology of school subjects during this era, and to the principle of psychologizing the subject matter on which it was based.

Thorndike's application of the universal principles of his connectionist theory of learning to academic content areas conflicted with Dewey's belief that the subject matter of the curriculum achieves its meaning and significance only with reference to the past, present, and potential experience of the learner, and that any psychology of school subjects must therefore "[a]bandon the notion of subject-matter as something fixed and ready-made in itself, outside the child's experience" (Dewey, 1902, p. 16). Thorndike sought to establish his psychology as a scientific enterprise by aspiring to conduct his research with an experimental precision characteristic of the natural sciences. This commitment—operationalized in the form of reducing the complexity of thought and reason to associations

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[1] Schwab (1978) distinguished between *substantive* and *syntactic structures* of a discipline. Summarizing the contrast, Shulman (1986) wrote,

> The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established. (p. 9)

between sensations and impulses—established a foundation for the neobehaviorism of Edward Tolman, Clark L. Hull, and Edwin R. Guthrie, and later the radical behaviorism of John B. Watson and B. F. Skinner (Kilpatrick, 1992). Collectively, these antimentalistic paradigms conflicted with Dewey's advocacy for psychologizing the subject matter, and thus contributed to the dormancy of his psychology of school subjects for half a century (Shulman, 1974).

In its motivation to establish a scientific approach to psychology, behaviorism supplied experimental methods and an analytic focus that would influence the trajectory of educational research from the 1950s through the 1970s. The radical behaviorists' critique of Titchenerian introspection as a tool for the qualitative analysis of conscious states—and as a valid source of psychological knowledge generally—combined with their strict focus on the objective data of behavior contributed to the broad appeal of classical experimental designs in early research on teaching and learning (Shulman, 1970). The actions exhibited by students and teachers in response to instructional, pedagogical, or environmental treatments comprised the primary analytical unit in this anti-cognitive genre of educational research.

Consistent with its behaviorist influences, research on teaching during its formative decades of the 1960s and 1970s assumed a predominantly *process-product* orientation (Dunkin & Biddle, 1974) in which investigators attributed desired learning outcomes to teachers' observable actions (Sherin et al., 2000). Within this research tradition scholars witnessed teachers employing a particular instructional practice, assessed student performance on tests of achievement or attitude, and via correlational analyses quantified the strength of the association between the teaching behavior and students' performance. Educational researchers adopting a process-product orientation conceptualized the teacher as a source of instructional treatments differentially associated with learning outcomes. It was the objective of this research paradigm to quantify these associations to identify the unique composition of teaching behaviors that maximize students' performance. Academic content was relevant in process-product studies only as a context variable limiting the generalizability of its statistically significant findings; subject-matter had neither a substantial influence on the nature of these findings nor the experimental and analytical methods from which they were constructed.

Around the time *A Nation at Risk: The Imperative for Educational Reform* was published in the United States by the National Commission of Excellence in Education in 1983, educational researchers in the U.S. were increasingly examining cognitive, social, and affective phenomena related to teaching instead of assuming teachers' behavior as their primary unit of analysis. An integral part of this shift involved characterizing the knowledge base that informs effective instruction. Consequently, propelled by the distressing conclusions of *A Nation at Risk* and sustained by the rising prominence of qualitative research methods introduced during the cognitive revolution in educational psychology, scholars in education devoted increased attention to understanding what teachers need to know and to examining the extent to which teachers' content and pedagogical knowledge informs their instructional actions. Although a departure from the anti-cognitivism of the behaviorist movement, the cognitive revolution in psychology inherited behaviorism's aspiration to discover universal mechanisms of learning. It is in this historical context that Shulman (1986) proposed a theoretical framework for teacher knowledge in response to characterizations of the knowledge base required for effective teaching that were polarized on a continuum ranging from strict subject matter knowledge to knowledge of pedagogy independent of any specific content domain.

Shulman's Conceptualization of Teacher Knowledge

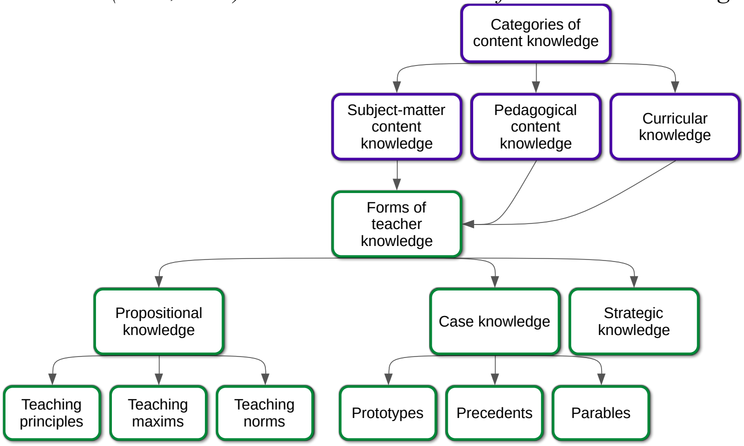
Shulman (1986) argued that research on teaching during the 1960s and 1970s, which inherited the methodological and epistemological characteristics of the then dominant process-product research paradigm, did not meaningfully attend to the subject matter being taught. He described this extensive disregard for subject matter among the established approaches to the study of teaching as the *missing paradigm problem*, and as a remedy proposed a theoretical framework for teacher knowledge that emphasized the foundational role of disciplinary content. Shulman did not encourage researchers' neglect of pedagogical knowledge in favor of subject matter knowledge but instead recognized that "to blend properly the two aspects of a teacher's capacities requires that we pay as much attention to the content aspects of teaching as we have recently devoted to the elements of the teaching process" (p. 8). Whereas Dewey's principle of psychologizing the subject matter was an admonition to educational psychologists for their general lack of consideration for students' experience in their learning of academic subjects, Shulman's missing paradigm was a response to the pervasive inattention to subject matter in early research on teaching.

Thorndike's efforts to describe general learning processes in terms of his connectionist psychology, combined with the appropriation of behaviorist methodologies by educational researchers anxious about their field's scientific respectability, contributed to the metamorphosis of Dewey's psychology of school subjects to the point that it abandoned a meaningful focus on academic content when research on teaching began in earnest in the 1960s. Shulman's argument for the reunification of psychological theory and the substantive and syntactic structures of academic disciplines was a timely revival of Dewey's principle of psychologizing the subject matter, contextualized for research on teaching and inspired by his opposition to the behaviorist influences that initially contributed to the decline of Dewey's psychology of school subjects in the early decades of the 20th century. Indeed, Shulman (2004) asserted, "the time has come for a renascence of a modern form of a psychology of school subjects" (p. 110). Shulman's (1986, 1987) conceptualization of teachers' knowledge base—particularly his notion of pedagogical content knowledge—embodied the integration of content and cognition for which Dewey (1902) so fervently advocated.

Shulman's theoretical framework provides a model for how content-related knowledge is organized in the minds of teachers. The constructs within this framework and their relations are illustrated in Figure 1. I discuss only the three categories of content knowledge Shulman distinguished: (a) subject matter content knowledge, (b) curricular knowledge, and (c) pedagogical content knowledge, emphasizing the last of these since PCK has been the most influential in more recent research on mathematics teachers' knowledge base.

Figure 1

Shulman's (1986, 1987) Theoretical Framework for Teacher Knowledge



Subject matter content knowledge "refers to the amount and organization of knowledge per se in the mind of the teacher" (Shulman, 1986, p. 9) and requires an understanding of both substantive and syntactic structures of a discipline (Schwab, 1978). Knowledge of the substantive structures of mathematics includes comprehension of its truths and techniques; whereas, knowledge of the syntactic structures involves understanding the means by which these truths are established (e.g., proof) and why such truths are worth knowing.

Shulman conceptualized teachers as the medium through which students experience the content of the curriculum. A teacher's *curricular knowledge* is therefore an essential component of their professional knowledge base. Curricular knowledge includes

> understandings about the curricular alternatives available for instruction … familiar[ity] with the curriculum materials

> under study by his or her students in other subjects they are studying at the same time … [and] familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them. (Shulman, 1986, p. 10)

Pedagogical content knowledge refers to the character of content knowledge needed for the practice of teaching. Shulman defined PCK as the knowledge of content that informs "the ways of representing and formulating the subject that make it comprehensible to others" (1986, p. 6). Additionally, Shulman (1987) described PCK as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 8). For Shulman, PCK involves transforming one's content knowledge into curricular material and pedagogical representations. This transformation consists of preparing and critically interpreting curricular materials, representing ideas in the form of analogies and metaphors, selecting appropriate teaching methods and models, and adapting pedagogical representations to the characteristics and needs of individual children (Shulman, 1987, p. 16). One does this effectively if the resulting pedagogical representations are accessible to students at a particular developmental level while accurately reflecting the normative characteristics of the academic discipline. Following Dewey, Shulman explained, "the psychologized subject matter is faithful to both of its constituents—the child and the curriculum—and that fidelity defines its intellectual honesty" (Shulman & Quinlan, 1996, p. 402).

Shulman (1987) further defined *pedagogical reasoning and action* as the process by which teachers develop, apply, and refine the various categories and forms of knowledge he distinguished (see Figure 1). Essential to pedagogical reasoning and action is a teacher's comprehension and transformation of the subject matter. *Comprehension* entails achieving a mature understanding of the curriculum—which involves knowing ideas in multiple ways and understanding their relation to other concepts within and across subjects—and discerning the generativity of students understanding the ideas being taught.

Transformation consists of preparing and critically interpreting curricular materials, representing ideas in the form of analogies and metaphors, selecting appropriate teaching methods and models, and adapting pedagogical representations to the characteristics and needs of individual children (Shulman, 1987, p.6).

The primary contribution of Shulman's conceptualization of teachers' knowledge base lies in his definition of PCK. Inspired by Dewey's (1902) proposal for educators to psychologize the subject matter, Shulman popularized the notion that pedagogical knowledge is shaped by one's comprehension of academic content. In doing so he exposed the conventional belief that effective pedagogies necessarily transcend disciplinary boundaries and established subject matter knowledge as a fundamental component of teachers' knowledge base.

Shulman argued for the relevance of his framework by citing the absence of theoretical tools available to support researchers' disciplined inquiry into the complexities of teacher understanding and transmission[2] of content knowledge (1986, p. 9). Moreover, he claimed that important questions like, "What are the domains and categories of content knowledge in the minds of teachers? How, for example, are content knowledge and general pedagogical knowledge related? … What are promising ways of enhancing acquisition and development of such knowledge?" (1986, p. 9) were not accessible using existing theoretical frameworks.

Mathematical Knowledge for Teaching

The research domain of *mathematical knowledge for teaching* (Thompson & Thompson, 1996) is an extension of Shulman's theoretical framework for teacher knowledge. Several researchers of mathematical knowledge for teaching

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[2] Shulman used the phrase "transmission of content knowledge" in the following context: "As we have begun to probe the complexities of teacher understanding and transmission of content knowledge, the need for a more coherent theoretical framework has become rapidly apparent" (1986, p. 9).

have based their elaborations, refinements, or extensions of PCK on interpretations of Shulman's work in ways that exhibit aspects of the behaviorist influences that contributed to the dissolution of Dewey's psychology of school subjects, and which inspired its modern articulation in Shulman's missing paradigm problem. In condensed form, these related interpretations are

1. PCK is simply the knowledge involved in *transforming* subject matter content into curricular material and pedagogical representations.
2. PCK is an *integration* of pedagogical and content knowledge.

These claims appear consistent with how PCK is described above. But as I argue in the following subsections, without attention to the construct's inspiration in Dewey's concept of psychologizing the subject matter, taken at face value the operationalization of these interpretations in MKT research has contributed to the production of empirical and theoretical results reminiscent of Thorndike's reductionism and reflective of the foundational commitments of the behavioral psychology it initiated.

Interpretation 1: PCK as a Transformation of Content Knowledge into Pedagogical Representations

Shulman's emphasis on the transformation of academic content into curricular material and pedagogical representations is strongly reflected in research that extends or refines his conception of PCK. Shulman and Quinlan (1996) explain that psychologizing the subject matter entails two related aspects: (1) an "analysis of the subject matter to find its essential features that can be rendered experientially meaningful to pupils" and (2) "the transformation of its mature and crystallized forms into representations that will be meaningful and educative to the child" (p. 402). Dewey (1902) emphasized the former of these aspects; he defined psychologizing the subject matter as a cognitive activity that results in one's capacity to provide the conditions for students to engage in the experiences necessary to stimulate their intellectual growth in a particular direction. The act of psychologizing the subject matter is fundamentally an accommodation of cognitive schemes that is motivated by teachers' consideration of the experiential basis for the substantive and syntactic structures of their academic subject. *Psychologizing the subject matter is thus a precondition for the transformation of teachers' disciplinary knowledge into meaningful and educative pedagogical representations*; it is not defined by this behavioral capacity.

In contrast to regarding the psychologization of subject matter knowledge as a process of cognitive reorganization (i.e., *accommodation*[3]), research that extends Shulman's conceptualization of teachers' knowledge base has maintained a predominant focus on the instructional or pedagogical behaviors afforded by psychologized subject matter schemes. This research has responded to Shulman's missing paradigm problem by first identifying the behavioral proficiencies that constitute effective teaching and then postulating the existence of subject-matter knowledge structures that support qualitatively distinct categories of these proficiencies (e.g., Hill et al., 2008). Knowledge categories are simply defined in terms of the behaviors they enable (i.e., knowledge of *X* is the knowledge required to do *X*). The cognitive mechanisms that facilitate the transformation of a teacher's subject matter knowledge for the demands of teaching are rarely made explicit in research on mathematics teachers' knowledge extending Shulman's pioneering work (Ball et al., 2008). By regarding PCK as the knowledge required to transform content knowledge into curricular material and pedagogical representations, MKT research has unsuspectingly inherited behaviorism's analytic focus on observable actions at the expense of discerning the characteristics of content-based knowledge structures from which these actions manifest (Tallman, 2021). The attention in MKT research on teachers' behavior has contributed to the development of a variety of knowledge constructs introduced to label and categorize behavior, not to explain it.

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[3] I define accommodation from a Piagetian perspective in a later section.

Interpretation 2: PCK as an Integration of Pedagogical and Content Knowledge

While researchers have strived in various ways to elaborate the notion of pedagogical content knowledge, they are generally united in their conception that PCK is a combination of content and pedagogical knowledge. In their systematic review of 60 empirical mathematics education research articles related to pedagogical content knowledge, Depaepe et al. (2013) found that despite widespread disagreement about its components and its static versus dynamic nature, "scholars agree that PCK connects at least two different forms of knowledge, i.e. content knowledge and pedagogical knowledge" (p. 15). According to the studies Depaepe and colleagues reviewed, what makes PCK "pedagogical" is that it results from an integration of pedagogical and content knowledge. Although many recent conceptualizations follow Shulman in defining PCK as an amalgam of knowledge of content and pedagogy, the relationship between these distinct types of professional knowledge, as well as the specifics of their synthesis, remains elusive. Seldom has research on PCK or MKT specified the nature of the conceptual activity that underlies the supposed integration of content and pedagogical knowledge, or identified the characteristics of content-based knowledge structures that when applied in instructional contexts suggest to an observer that the teacher's actions are based on an integration of qualitatively-distinct knowledge domains. Scholars assuming that PCK is an "amalgam" or "blending of content and pedagogy" (Shulman, 1987, p. 8) have tended to focus their work on either delineating the boundary between PCK and related forms of teachers' knowledge or specifying its diverse subcategories (Venkat & Alder, 2014). Once identified, these boundaries and subcategories have guided the development of instruments to measure PCK and have informed the design of pre- and in-service teacher education programs that seek to develop it (e.g., Hill & Ball, 2004; Hill et al., 2004).

Conceptualizing PCK as a unification of subject-matter and pedagogical knowledge encourages mathematics teacher educators to support pre-service teachers' construction of both knowledge domains independently—in content and methods courses respectively—and then to provide opportunities for them to apply both forms of knowledge in genuine teaching contexts to facilitate their integration. The recent popularization of "pedagogies of practice" (Grossman et al., 2009) and early practicum experiences—as exemplified by the common *UTeach* model for STEM teacher education in the United States—are representative of trends in pre-service teacher preparation guided by dynamic and integrative conceptions of PCK.

Instructing pre-service teachers in mathematical content and pedagogy separately with the expectation that these distinct knowledge domains will unify in the context of practice to support effective instruction is not without its limitations. Pedagogy, as the method and practice of teaching, describes a category of action. Pedagogical knowledge, then, refers to one's awareness of a repertoire of instructional practices and perhaps their association with the specific contexts in which their application has the greatest potential to improve students' academic performance. Leveraging mathematical contexts to instruct pre-service teachers in pedagogy is not sufficient to support their construction of PCK as Shulman conceptualized it. Moreover, this practice might result only in teachers' capacity to superficially and inflexibly enact specific behaviors in their uncritical efforts to imitate "evidence-based pedagogical practices."

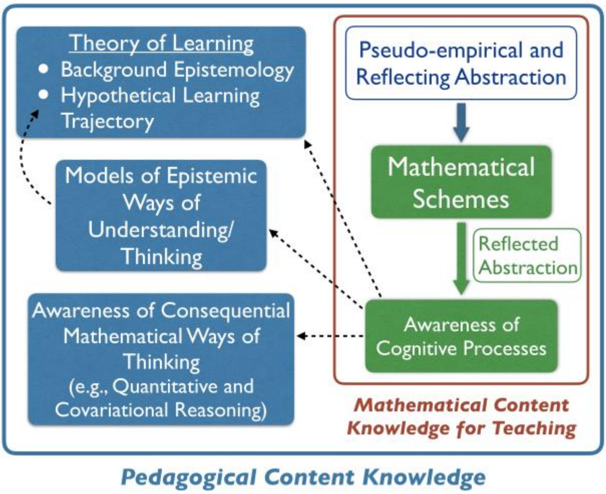
Abandoning a conception of PCK as an amalgam of content and pedagogical knowledge might encourage mathematics teacher educators to conceptualize their essential responsibility as supporting pre-service teachers' construction of mathematical knowledge structures with the precise characteristics that permit teachers' insight into the experiential basis of productive mathematical understandings and requisite ways of thinking. This (pedagogical) content knowledge might then facilitate teachers' natural implementation of pedagogical actions to purposefully engage students in the experiences necessary to promote their construction of desirable conceptions. In contrast to this approach, the common emphasis in mathematics teacher education on directly influencing teachers' instructional behaviors is at the very least maintained and at most encouraged by the assumption that PCK is an integration of subject matter and pedagogical knowledge.

Conceptions of PCK consistent with its Deweyan foundations are based on the notion that the psychologization of subject matter schemes is the essential process by which content knowledge is endowed with pedagogical utility (e.g., Tallman, 2015; Silverman & Thompson, 2008). From this perspective, subject matter knowledge, once psychologized, necessarily enables effective pedagogical action; the pedagogical nature of PCK is *not* the result of it being an additive combination of distinct knowledge domains. This conception of PCK prioritizes identifying the features of psychologized subject matter schemes that facilitate pedagogical efficacy, as well as understanding how these characteristics might be engendered in pre- and in-service teacher education. In service of this goal, I outline a constructivist theory of PCK that reconnects this construct with its historical motivation in Dewey's notion of psychologizing the subject matter.

Toward a Constructivist Theory of Pedagogical Content Knowledge

Uniting MKT research with the Deweyan philosophical foundations of PCK requires constructing a grounded theory for nature and development of the specific character of mathematical knowledge that positions teachers to enact effective pedagogies in the service of supporting students' construction of productive meanings. In other words, understanding the process by which teachers psychologize the subject matter of mathematics is an urgent priority for MKT research. Figure 2 displays various constructs and the relations between them that comprise the components of a provisional theoretical framework—grounded in radical constructivism and Piaget's genetic epistemology—that seeks to address this need.

Figure 2
Theoretical Framework for Pedagogical Content Knowledge



Reflecting and Reflected Abstraction

Schemes and Equilibration

Piaget generally defined a cognitive structure, or *scheme*, as “the structure or organization of actions as they are transferred or generalized by repetition in similar or analogous circumstances” (Piaget & Inhelder, 1969, p. 4). Schemes serve to organize the individual’s reality and impose order on their experiences by equipping the individual with the conceptual tools to systematically act on their environment and anticipate particular outcomes. To explain the genesis and refinement of cognitive schemes, Piaget elaborated the concept of equilibration, the mechanisms of which are assimilation and accommodation. Briefly defined, *equilibration* is the self-regulatory process by which an individual actively compensates for external disturbances (Piaget & Inhelder, 1969); *assimilation* is the process whereby a subject incorporates experiences into existing cognitive structures, and thus consists of the meanings the subject holds; and *accommodation* entails the modification of an individual’s cognitive schemes to enable their assimilation of novel experiences (Piaget, 1977). Assimilation and accommodation, and thus equilibration, rely heavily on the notion of abstraction, of which Piaget distinguished five varieties: *empirical*, *pseudo-empirical*, *reflecting*, *reflected*, and *metareflection* (Piaget, 2001). Piaget explained that higher forms of knowledge originate from abstractions of the subject's actions, and the results of applying them in specific contexts. As demonstrated in the following discussion, the nature of the cognitive schemes that characterize mathematical knowledge are organizations of internalized (mental) actions constructed through the process of reflecting abstraction and refined through further projection of reasoning to the reflected level of thought.

Reflecting and Reflected Abstraction

Central to the theoretical framework for PCK depicted in Figure 2 is teachers' construction of mathematical content knowledge through reflecting and reflected abstractions (Piaget, 2001). *Reflecting abstraction* involves the reconstruction on a higher cognitive level of the coordination of actions from a lower level and results in the development of cognitive structures, or schemes, at the level of operative thought[4] (Chapman, 1988; Piaget, 1971). The resultant cognitive schemes are organizations of internalized actions and operations. Reflecting abstraction is thus an abstraction of actions that occurs in three phases: (1) the *differentiation* of an action from the effect of the action, (2) the *projection* of the action from the level of material action to the level of representation, and (3) the *reorganization* that occurs on the level of representation of the action projected from the level of material action.[5]

[4] Schemes constructed at the level of operative thought are comprised of reversible mental actions (i.e., operations) that can be applied to a generic class of objects without regard for an initial state. The operative aspect of thought relates to how a knowing subject structures their experiences by assimilation of figurative material (Müller, 2009, p. 223). Knowing subjects possess the figurative functions of perception, imitation, and mental imagery that produce the material for assimilation, which become the objects on which operative schemes perform actions and transformations.

[5] See Ellis et al. (in press) for a thorough discussion of reflecting abstraction.

When an individual becomes cognizant of the coordinated actions projected to the level of representation through reflecting abstraction, we say that the resulting abstraction becomes *reflected*, or that the organization of internalized actions exists at the *reflected level of thought* (i.e., the "plane of thematization" [Piaget, 2001, p. 51] where projected actions become the objects of mental operations). The reflected level of thought is a higher cognitive plane to which one's reasoning (as opposed to actions as in the case of reflecting abstraction) is projected. Piaget's distinction between reflecting and reflected abstraction suggests that conscious knowledge is not a byproduct of reflecting abstraction alone. Whereas empirical, pseudo-empirical, and reflecting abstractions are all constructive processes, reflected abstraction is a product of reflecting on the projected actions from previous[6] reflecting abstractions, which results in the knower's awareness of these internalized actions. Thus, reflected abstractions enable an individual to explicitly formulate the results of prior reflecting abstractions. Reflected abstraction, then, describes a scheme constructed at the reflected level of thought through reflection on the results of prior reflecting abstractions. These reflected schemes possess an essential characteristic (cognizance of projected action coordinations, decontextualized and applicable to a generalized class of objects) that endows the knower with the capacity to explicitly formulate meanings and strategically apply them in a a range of novel contexts.

Reflected Abstraction and the Psychologization of Subject Matter Schemes

Actions are performed in consonance with an operational goal structure; all behavior serves to facilitate the attainment of some goal organized within a fluid hierarchy where the

[6] Regarding the décalage, or temporal delay, between reflecting and reflected abstraction, Piaget (2001) explained, "The subject becomes conscious of the result of his acts—which requires a simple static read-off—before becoming conscious of their mechanism and their exact unfolding—which requires the reconstruction of a process" (p. 191).

accomplishment of subordinate goals facilitates the achievement of superordinate ones. In the case of mathematics instruction, teachers implement pedagogical practices for the purpose of supporting students' construction of meaning. These actions can be epistemologically justified or not. That is, a teacher can be capable of rationalizing their pedagogical actions with an explicit appeal to the cognitive activity they expect to engender, or they can act in ways that uncritically conform to their image of "best teaching practices." The former orientation, facilitated by the construction of reflected mathematical schemes, enables a teacher's purposeful, flexible, and strategic enactment of pedagogical actions in a way that is responsive to their model of students' current and developing conceptions. The awareness of the internalized actions and operations organized within one's mathematical schemes that results from having constructed meanings at the reflected level of thought is thus the essential characteristic of a teacher's mathematical knowledge that facilitates their pedagogical potential (Tallman, 2015; Tallman & Weaver, 2018; Silverman & Thompson, 2008; Thompson, 1985, p. 222-223). Discussing the need for teachers to transform *key developmental understandings*[7] (KDUs; Simon, 2006) into *key pedagogical understandings*, Silverman and Thompson (2008) similarly described the relationship between teachers' mathematical conceptions and their pedagogical actions:

> Teachers who develop KDUs of particular mathematical ideas can do impressive mathematics with respect to those ideas, but it is not necessarily true that their understandings are powerful pedagogically. It is possible for a teacher to have a KDU and be unaware of its utility as a theme around which productive classroom conversations can be organized. Developing MKT then involves transforming these personal KDUs of a particular mathematical concept to an understanding of (1) how this KDU could empower their students' learning of related ideas and (2) actions a

[7] Key developmental understandings have the potential to lend coherence to several concepts within the mathematics curriculum and to advance students' ability to interpret and reason about a range of ideas.

> teacher might take to support students’ development of it and reasons why those actions might work. (p. 502)

For the reasons previously outlined, reflected abstraction is fundamental to this transformative process. As indicated in Figure 2, the cognizance that results from a teacher having constructed mathematical meanings at the reflected level of thought can motivate their attention to epistemology, necessitate their construction of epistemic ways of understanding (Liang, 2020), and promote their consideration of consequential ways of thinking and reasoning (Tallman & Frank, 2020). I discuss these implications of reflected mathematical schemes in the following subsections.

Background Epistemology and Hypothetical Learning Trajectory

In its most general description, a mathematics teacher is responsible for providing opportunities for students to engage in the conceptual activity required for their construction of productive meanings (Thompson, 2013). Accomplishing this goal demands that the teacher’s actions be deliberately informed by an understanding of the functional mechanisms of mathematics learning so that these mechanisms can be purposefully engendered through instruction. This essential obligation cannot be fulfilled if a teacher is aware only of the behavioral capacities that result from applying their mathematical schemes in specific contexts. A teacher operating with this type of awareness primarily seeks to support students in becoming fluent at engaging in the sequence of actions by which they can successfully complete routine tasks. Alternatively, a teacher cognizant of the mental actions and operations that comprise their own mathematical schemes is positioned to reflect on the conceptual process by which students might construct similar conceptions. This implication of a teacher’s awareness of the contents of their subject matter knowledge is captured in the “Background epistemology” component of Figure 2.

Effective instruction additionally necessitates that the teacher’s actions be informed by an image of how students might develop particular mathematical conceptions. Constructing this image entails (1) clarifying what it means to understand a specific mathematical idea; (2) discerning the actions, abstractions, and generalizations in which a student must engage to construct this meaning; and (3) designing curricular artifacts to provoke such cognitive activity. These three components are consistent with Simon's (1995) articulation of a hypothetical learning trajectory (HLT) and Thompson's (2008) description of conceptual analysis. Engaging in conceptual analysis to construct an HLT for a mathematical idea relies upon one's awareness of the mental actions and conceptual operations that constitute their scheme for the idea. Moreover, the second and third components of an HLT necessitate a teacher's consideration of students' potential experiences as they engage with and progress through a curriculum designed to promote desired conceptual activity. These two components are based on an explication of the experiential basis of targeted mathematical conceptions, which comprises the first component. Conceptual analysis is therefore the means by which one psychologizes mathematical subject matter, and as previously emphasized, psychologizing the subject matter is the process that endows content knowledge with pedagogical efficacy. Reflected abstraction is the essential cognitive mechanism that facilitates a teacher's psychologization of mathematical subject matter and is thus foundational to teachers' construction of PCK.

Models of Epistemic Ways of Understanding/Thinking

Another potential implication of a teacher having engaged in reflected abstraction is that the ensuing awareness of the conceptual contents of their mathematical schemes motivates and enables their construction of second-order models[8] (Steffe et al., 1983) of students' mathematical meanings. Epistemic ways of understanding are generalizations of these second-order

[8] Thompson (2000) distinguished first- and second-order observers as follows: "first-order observers address what someone understands, while second-order observers address what *they* understand about what the other person *could* understand" (p. 303, italics in original).

models, the construction of which requires a teacher to maintain a particular orientation toward their interactions with students. Specifically, a teacher must be committed to inferring the conceptual operations that explain students' language and actions and to engage them in experiences that make such insights possible, robust, and viable. Such strategic interaction is informed by one's awareness of the mental actions and operations that comprise their mathematical schemes, as this cognizance serves as a basis of comparison for how students might understand a mathematical idea.

Models of epistemic ways of understanding additionally empower a teacher to design instruction with an anticipation of how their conceptual goals for students' learning might result from a specific progression of cognitive experiences. This capacity resonates with how Dewey (1902) described an educator's essential commitment: a teacher "is concerned with the subject-matter of the science as *representing a given stage and phase of the development of experience*" (p. 30, 1902, emphasis in original). Constructing epistemic ways of understanding are thus necessary to developing hypothetical learning trajectories, particularly their second and third components.

Awareness of Consequential Mathematical Ways of Thinking

Harel (2008a) articulated a distinction between mathematical ways of understanding and ways of thinking. Thompson and Harel refined these theoretical constructs in relation to Piaget's notions of scheme and assimilation. They described one's *way of understanding* a concept as constituting their scheme for the concept, and they characterized *ways of thinking* as habitual anticipations of specific meanings while engaged in the act of reasoning (Thompson et al., 2014, p. 12). Quantitative reasoning, for example, is a way of thinking. An individual's inclination to conceptualize situations in terms of quantities and quantitative relationships is one that can be productively applied to make sense of several mathematical ideas. An individual who maintains an orientation across a variety of mathematical domains to identify measurable attributes of objects and to define relationships between them possesses the habitual anticipation to leverage this meaning while reasoning about novel problems or situations. Similarly, *covariational reasoning* (Carlson et al., 2002; Thompson & Carlson, 2017) is an important way of thinking mathematically; inclinations to conceptualize variation in a *smooth-continuous*[9] manner (Castillo-Garsow, 2010, 2012; Thompson & Carlson, 2017) can support students' capacity to reason productively about mathematical objects and representations.

Harel's *duality principle* suggests the process by which ways of thinking might mature: "Students [or teachers] develop ways of thinking through the production of ways of understanding, and, conversely, the ways of understanding they produce are impacted by the ways of thinking they possess" (Harel, 2008b, p. 899). The cognitive processes that underlie the duality principle, however, are insufficiently specified to inform interventions to advance teachers' ways of thinking, or to make them cognizant of their critical role in supporting students' construction of coherent ways of understanding. Piaget's notion of reflected abstraction might attenuate this limitation. If a teacher can develop productive ways of understanding specific mathematical ideas and become consciously aware of the mental actions and operations that constitute them, then they might be positioned to abstract common features of their mathematical schemes and recognize how particular ways of thinking contributed to their construction and/or facilitate their application. This kind of reflective activity is indispensable for teachers because it equips them with an image of how they might

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[9] Smooth continuous variational reasoning entails a conception of variation supported by images of fictive motion wherein the varying quantity's measure passes through a compact interval of values—the quantity's measure assumes all values within an interval by virtue of conceiving its variation smoothly and continuously through the interval containing these intermediary values (Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). Smooth continuous covariational reasoning describes the conceptual activity involved in uniting the simultaneous variation of two quantities' measures, each conceived as varying in a smooth continuous manner.

leverage specific ways of thinking in their teaching to support students' construction of coherent mathematical meanings.

Conclusion

I have argued for the need to reconnect MKT research to the Deweyan philosophical inspiration of Shulman's notion of pedagogical content knowledge. The behavioral emphasis of contemporary MKT research reflecting two interpretations of Shulman's work is consistent with trends in educational psychology that contributed to the quiescence of Dewey's central ideas for much of the 20th century. These ideas, particularly Dewey's principle of psychologizing the subject matter, motivated Shulman's articulation of the missing paradigm problem—a version of Dewey's proposal for the experience of the learner to authenticate the psychological study of academic subjects—of which the profusion of ensuing research on teachers' knowledge was an intended but generally insufficient remedy. As an alternative, I have argued that conceptualizing MKT in terms of the characteristics of content-based (operative) schemes that enable effective pedagogical action is a more compelling response to Shulman's missing paradigm. I have proposed a provisional theoretical framework for pedagogical content knowledge that specifies the process and potential implications of a teacher having psychologized mathematics subject matter by constructing operative mathematical schemes grounded in reflecting and reflected abstractions. This framework leverages Piaget's genetic epistemology to extend Shulman's initial conception of PCK while remaining coherent with its Deweyan foundations. I conclude by emphasizing what makes this conception of PCK *pedagogical*.

Pedagogy emerges from a teacher's application of content knowledge to achieve their instructional goals in the context of practice, and is for this reason a behavioral expression of the nature and character of one's PCK, not a category of it. Pedagogical content knowledge *is* content knowledge with characteristics that endow it with pedagogical utility; the pedagogical character of PCK derives from the specific ways a teacher's knowledge of subject matter informs their enactment of effective instructional actions. So rather than conceive the pedagogical nature of PCK as the result of integrating content and pedagogical knowledge, or as a consequence of the experimental context from which grounded theories for PCK were developed (e.g., "practice-based" approaches [Ball et al., 2001]), I have appealed to the Deweyan theoretical foundations of PCK to argue that pedagogical content knowledge is more productively conceived as *psychologized content knowledge*. The pedagogical capacities PCK enables are afforded by having connected the content of an academic subject with experiential basis of its substantive and syntactic structures. An essential characteristic of a mathematics teacher's content knowledge with pedagogical implications is the extent to which they are cognizant of the mental actions and operations that comprise the meanings they design their instruction to support, which results from having constructed and refined mathematical schemes through reflecting and reflected abstractions (Tallman & Weaver, 2018). Engaging in reflected abstraction is necessary to establish the connection between the content of mathematical subject matter and its origin in the cognitive experience of the learner. With this connection established, a teacher is positioned to strategically and responsively enact pedagogies to engage students in the precise experiences required for their construction of targeted mathematical understandings. Reflected abstraction is thus the essential cognitive mechanism by which mathematical content knowledge is psychologized, resulting in its potential to be applied in pedagogically efficacious ways.

A more prominent goal of MKT research should therefore be to determine how teacher educators might support pre- and in-service teachers' construction of mathematical schemes that facilitate their capacity to enact effective pedagogies, rather than to maintain the traditional practice of teaching content and pedagogy separately with the expectation that these distinct knowledge domains will cohere in the context of practice to support high quality instruction. As previously noted, many scholars who have extended Shulman's work have aspired to identify categories, or empirically distinguishable subdomains, of PCK. Examining the extent to which the competencies suggested by (or sometimes defined as) these subdomains might be achieved as a byproduct of having constructed content knowledge in a way that maintains the characteristics that endow it with pedagogical utility is an important area of future research into teachers' PCK. This research agenda could enable future scholarship on mathematics teacher knowledge to no longer be restricted by interpretations of Shulman's work that once stimulated an eruption of research activity, but which are now imposing barriers to achieving the insights necessary for explaining teachers' instructional actions and for developing theory-informed innovations to systematically improve mathematics teachers' professional knowledge base.

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