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Intellectual Need, Covariational Reasoning, and Function: Freeing the Horse from the Cart

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Supporting young students in developing meanings for the set-theoretic function definition is emphasized in 6th-12th grade curricula around the world. In prior work, we have highlighted how covariational reasoning supports college students in constructing relationships that afford considering mathematical properties important for a set-theoretic definition. In this paper, we show how such reasoning can provide similar affordances for younger students by presenting one sixth grade student's, Ariana's, sense-making. To characterize Ariana's sense-making related to her quantitative reasoning in contextual situations, we build on Harel's work to articulate the constructs of situational intellectual need and situational epistemological justification. We highlight how Ariana's covariational reasoning supported her development of a situational epistemological justification, which included a structure entailing numerous quantitative relationships. We also highlight how this epistemological justification supported her work representing conceived relationships graphically and making determinations regarding properties of a set theoretic function definition. However, we characterize that Ariana constructed functional and non-functional relationships alike; determinations regarding properties of function were spurned by teacher-researcher prompts rather than an intellectual need experienced as she conceived such relationships. Through this analysis, we build an anti-deficit story of Ariana's sense-making that leads us to call into question the value of focusing on the set-theoretic definition of function early in students' experiences.

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> *A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (Thompson & Carlson, 2017, p. 444)*

Curricular standards in the U.S. and elsewhere emphasize the importance of middle and high school students learning a set-theoretic function definition (Ayalon & Wilkie, 2019; National Governors Association, 2010). For instance, in the U.S. students are expected to learn a set-theoretic function definition in 8th-grade (National Governors Association, 2010). High-stakes U.S. state tests require students to identify whether relationships presented as tables (e.g., New York State Education Department, 2022; North Carolina Department of Public Instruction, 2019) and as graphs (e.g., Massachusetts Department of Elementary and Secondary Education, 2022; Ohio Department of Education, 2022) represent functions. After students are introduced to a set-theoretic function definition, their mathematical experiences are largely restricted to exploring various function classes (i.e., linear, quadratic, exponential). Non-functional relationships (per traditional textbook definitions of function) are largely absent from U.S. curricula after the introduction of a set-theoretic function definition.

Despite the importance of the set-theoretic definition of function in school mathematics, most research points to students not understanding the definition in ways compatible with mathematician or educator intentions (e.g., Breidenbach et al., 1992; Even, 1990; Martinez-Planell & Gaisman, 2012; Moore et al., 2019a). Whereas some researchers have designed interventions to promote more productive meanings for the function definition for high school (e.g., Dubinsky & Wilson, 2013) or college students (e.g., McCulloch et al., 2019; McCulloch et al., 2022), in this paper we take a different approach. We call for a de-emphasis on the function definition in school mathematics altogether in favor of developing students quantitative and covariational reasoning. We ground this argument in our prior research and the results presented in this paper.

Extending the Horse and Cart Metaphor

In our previous work (Paoletti & Moore, 2018), we argued a conception of function rooted in covariation, as described in the opening quote, can provide students with meanings for quantitative relationships that support them in making determinations about the mathematical properties important for a set-theoretic function definition (e.g., univalence). In that previous work we illustrated the productivity of such a conception using a case of an undergraduate student, Arya, constructing a quantitatively sophisticated image of a situation. Arya then leveraged this image to determine if various situational relationships (some represented graphically and others only imagined) represented functions by considering the set-theoretic function properties she had previously learned. We contended that Arya's covariational reasoning provided her with a metaphorical horse that she could use to pull the metaphorical cart that is a formal set-theoretic function definition.

In this paper, we return to the horse and cart metaphor to build on and extend our previous argument by constructing an anti-deficit story (Adiredja, 2019; Adiredja & Louie, 2020; Adiredja & Zandieh, 2020) describing the productive sense-making of Ariana, a Latina 6th- grade student (approximately 11-years old). To do this, we first extend Harel's (2008, 2018a, 2018b) prior work to the domain of students' quantitative reasoning to introduce the constructs of situational intellectual need and situational epistemological justification.[1] We highlight how Ariana experienced a situational intellectual need that supported her in constructing a situational epistemological justification. Ariana leveraged her situational epistemological justification as she developed meanings for graphs as representing emergent traces representing covariational

[1] We note our use of "situational" refers to the situation, context, or physical phenomena students are making sense of and is not intended to refer to situated perspectives (e.g., Lave & Wenger, 1991).

relationships (Moore & Thompson, 2015; Moore, 2021). Further, she leveraged this justification as she addressed questions related to properties of the set-theoretic definition of function in ways compatible with Arya (Paoletti & Moore, 2018), despite never having been introduced to a set-theoretic definition of function.

Although Ariana (a 6th-grade student) was capable of engaging in reasoning compatible with Arya (an undergraduate student), we highlight how Ariana did not experience any intellectual need for differentiating between functional and non-functional relationships. By connecting to research on students' and teachers' meanings for function, we argue that a potential lack of intellectual need raises questions regarding the importance of emphasizing a set-theoretic function definition that over-privileges the importance of univalence (the property that for each element in the domain there is a unique element in the range) as required by pre-college mathematics curriculum standards. That is, we argue for freeing the metaphorical horse (i.e., constructing quantitative relationships) from the cart (i.e., a formal function definition), with the cart only being brought in when students experience some intellectual need for it (e.g., in analysis and exploring formal properties of integration and differentiation).

Intellectual Need, Epistemological Justifications, and Constructing Quantitative Relationships

Harel (2008, 2018a, 2018b) included the constructs of intellectual need and epistemological justification as part of his framework for DNR-based instruction. We use two criteria to characterize a student as experiencing an intellectual need.[2] First, the student must have an experience in which their current ways of operating (e.g., mathematics knowledge) does not result in assimilation and establishing a state of equilibrium, thus resulting in a state of perturbation. Second, a researcher claiming

[2] We note that intellectual need, as defined in the broadest sense, can stem from the enactment of mathematical or non-mathematical schemes (e.g., affective schemes). For the purpose of this paper, we focus on mathematical schemes.

that a student is experiencing an intellectual need implies that the researcher perceives meanings to be within the student's zone of proximal development that could resolve the state of perturbation (Weinberg et al., 2023).[3] If, on the other hand, the student experiences a perturbation such that the meanings necessary for accommodation are outside of their zone of proximal development, the student's perturbation is better characterized as associated with a state of confusion rather than a state of intellectual need (Weinberg et al., 2023).

If the student is able to resolve an intellectual need via the creation of new mathematical knowledge, and is aware of how the new knowledge resolves the perturbation, Harel (2008) characterizes the resulting awareness as the student's *epistemological justification*. Using the context of complex numbers, Harel (2018a) exemplified the difference between students developing new knowledge without and with an awareness of how a perturbation is resolved. He described how his college students had been taught about complex numbers, and how they could operate on complex numbers, without ever having experienced any intellectual need for such numbers. He also described an approach to introducing complex numbers that started with students experiencing a perturbation. This perturbation could be resolved by creating a definition for complex numbers. The students' prior knowledge around complex numbers was not grounded in any epistemological justification, whereas the students experiencing Harel's approach could generate a sentential epistemological justification, which results from understanding the need for a definition, axiom, or proposition for complex numbers.

Emphasizing the importance of intellectual need and epistemological justification, Harel (2018a) advised that instruction focused on rigor (e.g., formal mathematical definitions) in absence of intellectual need for that rigor creates situations in which students feel like "aliens in knowledge

[3] As Weinberg, Tallman, & Jones (2023) clarify, the particular meanings a researcher perceives to be within a student's zone of proximal development are the *for* when a researcher describes a student experiencing *an intellectual need for* a particular idea.

construction" (p. 38). In such an absence, students are unlikely to value rigor and, relatedly, unlikely to construct an associated epistemological justification rooted in their mathematical meanings.

In addition to sentential epistemological justification, Harel (2018a, 2018b) has described other forms of epistemological justifications including understanding aspects of the process of proving (apodictic epistemological justification) and understanding underlying reasons for how a proof or justification came into being (meta epistemological justifications). Across all of Harel's characterizations of intellectual need and epistemological justification, he emphasizes the importance of students experiencing perturbations that they resolve via the construction of some new mathematical knowledge.

Due to his focus on new mathematical knowledge, Harel does not explicitly focus on student's meanings for situations that may support the generation of new mathematical knowledge. We add to the types of intellectual need and epistemological justifications by describing a *situational intellectual need* and *situational epistemological justification*. We characterize a student as experiencing a situational intellectual need when she experiences a (possibly minor) perturbation as she conceives a novel "real-world" situation and subsequently sets the goal-oriented activity of making sense of and mathematizing that situation via a cyclical process of constructing quantities and their relationships. Johnson's (2023) description of an intellectual need for relationships, "a need to explain how elements work together, as in a system" (p. 30) falls within our description of a situational intellectual need.

We characterize the student as creating a situational epistemological justification when she resolves this perturbation by leveraging, and potentially reorganizing, her existing schemes and operations in a way that provides her with both an understanding of the situation and an awareness of the underlying quantities and relationships between quantities in the situation. Our characterization of intellectual need and epistemological justification are less stringent than Harel's use; we do not require the construction of knowledge in the form of entirely new schemes and operations. However, we underscore that students re-constructing or reorganizing previously constructed (quantitative) schemes and operations in a novel situation is effortful, as well as critical for the construction of mathematical concepts (e.g., Steffe & Thompson, 2000).

When characterizing intellectual need and epistemological justifications, the researcher's goal should be to explore and explain the student's purposeful sense-making in the context as the student understand it, which is consistent with an anti-deficit perspective (Adiredja, 2019), and pursuing a humanized, equitable education via attention to students' mathematics (Ellis, 2022; Hackenberg, 2010). Certain situations or tasks may elicit an intellectual need for some students and not for other students. When a task does not elicit an intellectual need, researchers and teachers need to consider why in relation to the student's current meanings; the notion of intellectual need does not exist independent of situating it in the context of a student's extant mathematics in combination with an instructor's or researcher's targeted meanings in working with that student (Weinberg et al., 2023). Such reflections can support the design (or re-design) of tasks that can be further implemented.

Situational Intellectual Need, Epistemological Justifications, and Emergent Thinking

As a backdrop to illustrate the notions of situational intellectual need and epistemological justification, we use the *Faucet Task* (and student work on this task in subsequent sections), which we have implemented in both research and instructional settings. As situational intellectual need requires a student to conceive a situation and set the goal-oriented activity of making sense of and mathematizing a situation, it is important to use experientially real situations (Gravemeijer & Doorman, 1999). Experientially real situations give students the opportunity to construct quantities and their relationships (e.g., Johnson et al., 2020; Thompson & Carlson, 2017). As students have experiences with running water and faucets, we assume the *Faucet Task* is experientially real to them.

Creating Situational Intellectual Need

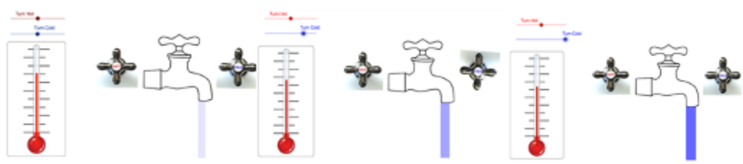
To support students in connecting the Faucet Task to an experientially real situation, we have them explore a Geogebra applet that allows them to turn hot and cold knobs. Turning each knob results in changes both to the width of the rectangle below the faucet, which indicates changes in the amount of flowing water, and to the color of the rectangle, which indicates changes in the temperature of the water (see https://www.geogebra.org/m/rdxkrwek and Figure 1). When implementing the task, we first have students identify and describe quantities in the situation they could measure (e.g., amount of turns of either knob, water temperature, amount of water) to explore if they are understanding quantities in ways compatible with our intentions. Intending to support the students in experiencing a situational intellectual need that leads to their mathematizing the situation, we present four tasks, each beginning with both knobs turned halfway on (Figure 1, left). We ask students to predict how temperature and amount of water vary *from this initial state* as (A) the cold knob is turned to all the way on, (B) the cold knob is turned to all the way off, (C) the hot knob is turned to all the way on, and (D) the hot knob is turned to all the way off. In each case, we ask students to provide reasons for their prediction prior to using the applet to check if their prediction is viable. For example, a student addressing prompt B may argue that since the cold knob is being turned off, the amount of water will decrease, and there will be less cold water so the water temperature will increase.

Building a Situational Epistemological Justification

When students observe the quantities changing in a way other than their prediction, we ask them to consider why, in a faucet situation, the quantity did not do what they anticipated. For example, it is not uncommon for students to predict increasing the cold water will cause both the amount of water and temperature to increase. However, after observing the temperature decreasing, students have the opportunity to re-construct and make accommodations to their meanings for the relationships between quantities in the situation. For example, they may consider how adding cold water will increase the relative proportion of cold to hot water from the starting combination of equal amounts hot and cold water, thereby decreasing the water temperature. Such reasoning can provide the basis for an evolving situational epistemological justification such that they begin to develop an awareness of the underlying quantities and relationships between quantities in the situation.[4]

Figure 1

Screenshots of the Faucet Task for Scenario (A) the cold-water knob being turned on from its initial state.



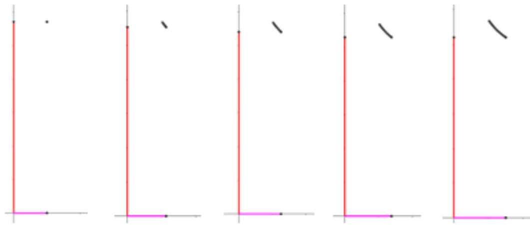
Leveraging a Situational Epistemological Justification to Develop Graphing Meanings

We next use the *Faucet Task* to support students in leveraging the situational epistemological justifications they developed in the above activity to build towards a conception for graphs termed *emergent graphical shape thinking* (Moore, 2021; Moore & Thompson, 2015). Drawing on descriptions of covariational reasoning (see Thompson & Carlson, 2017), Moore and Thompson (2015) described emergent thinking as conceiving a graph simultaneously in terms of "what is made (a trace) and how it is made (covariation)" (p. 785). Critical to such a conception is a student conceiving of a graph in terms of a progressive trace constituted by a point's movement dictated by the covarying quantities' magnitudes. Hence the resulting graph is an emergent result of that covariation (see Figure 2). Such reasoning requires explicit bridging of students' meanings for

[4] We note it is common for students to share that they have thought of this task between sessions while cleaning dishes at home or using a school sink. Such experiences provide further opportunities for them to develop situational epistemological justifications.

objects in a coordinate system (e.g., segments representing quantities' magnitudes) and the covarying quantities in a situation (Gantt et al., 2023; Paoletti et al., 2023).

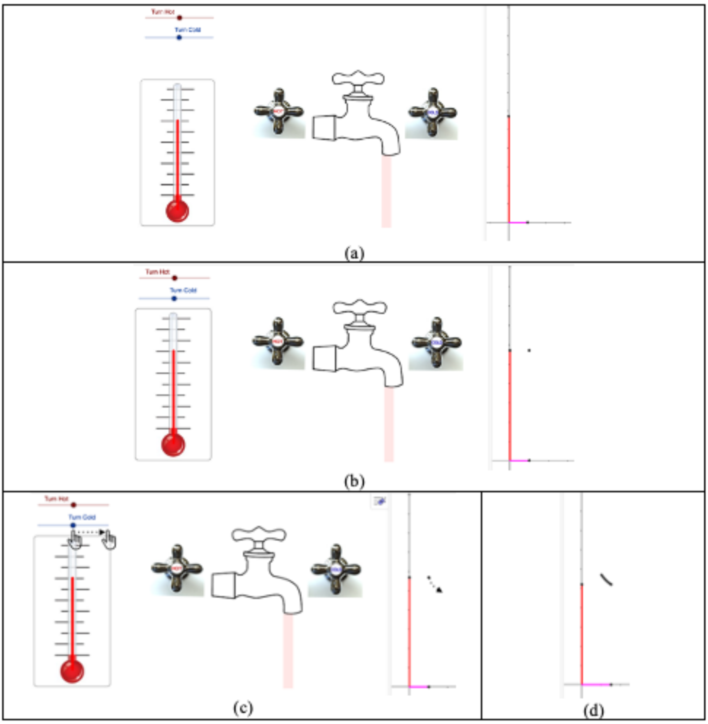
Figure 2
Several Static Instances of the Emergent Trace Representing Amount of Water and Temperature Covarying as the Cold Knob is Turned On



With the goal of supporting the students' development of emergent graphical shape thinking, after the task sequence described above, students engage with a series of applets, each presenting the original situation with additional mathematical objects. The first of these applets presents temperature and amount-of-water magnitudes on a vertical and horizontal axis, respectively (Figure 3a). The next applet presents a point in the coordinate system simultaneously moving in accordance with each segment's magnitude (Figure 3b). In the third applet, the 'trace' feature of Geogebra is used to have the dynamic point, representing both quantities' magnitudes, leave a trace that produces a record of the movement of the point.

For each of these three applets, we again have students predict, test, and observe what happens for Scenarios (A)–(D). Students can leverage their situational epistemological justification as they describe how different objects in the coordinate system change based on their meanings for the situation. For example, a student may anticipate that when the cold water is turned on, the amount of water will increase and the temperature will decrease. That student can also anticipate these changes in the quantities would be represented by the pink segment on the horizontal axis getting longer and the red segment on the vertical axis getting shorter. The student may then argue that the point will move diagonally down

Figure 3
Screenshots of the Applet Showing Corredponding Magnitudes, Points, and Trace



Note. The applet (a) with temperature magnitude on the vertical axis and amount of water magnitude on the horizontal axis (b) with the point shown, (c) showing the movement of the knob and point as the cold is turned on. (d) The resulting emergent trace from (c).

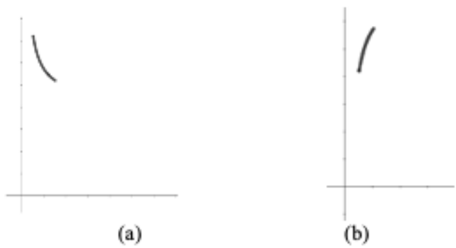
and to the right because of these changes in the two segments (see Figure 3c for a trace for Scenario (A)). Further, we note this series of applets and prompts can create additional opportunities for students to re-construct a situational epistemological justification. Each applet presents a new object for the student to consider, which can result in the student setting the goal-oriented activity of making sense of and mathematizing that object in relation to their previous activity. When objects do not behave as predicted, students have repeated opportunities to re-conceive the quantities and their relationships in the situation (and in the graph). Hence students have additional possibilities to re-conceive or strengthen their situational epistemological justification.

Relevant to students' emergent reasoning, Paoletti and Moore (2017) characterized that reasoning about the same graph as being traced in multiple directions was a strong indication of emergent graphical shape thinking. Hence, in the last part of the *Faucet Task*, we ask students to interpret what situations may have created novel completed graphs (see Figure 4 for examples). By asking students to interpret such graphs, we intend for them to experience another round of situational intellectual need as they set the goal of interpreting mathematical representations in relation to situational quantities and relationships between the quantities. The students can reconcile this intellectual need by drawing on and accommodating their previously constructed situational epistemological justifications. Namely, students may re-construct specific, and maybe several, quantitative structures to interpret given graphs as tracing in one, and maybe several, directions. For example, a student may interpret the graph in Figure 4a as tracing from left-to-right, arguing temperature is decreasing while amount of water is increasing. The student may conclude turning the cold knob on would produce this graph. A student may also interpret the graph as tracing from right-to-left, arguing temperature is increasing while the amount of water decreasing. With this interpretation the student may conclude turning the cold knob off would produce this graph.

Anti-deficit Perspective, Radical Constructivism, and Teaching Experiments

In this section, we characterize our understanding of adopting an anti-deficit perspective. We then describe how we view radical constructivist teaching experiments (Steffe & Thompson, 2000) as a viable tool researchers can use to develop anti-deficit stories of students' productive sense-making. An

Figure 4
Two Examples of Completed Graphs in the Faucet Task



anti-deficit perspective is a theoretical orientation researchers can use to examine students' mathematical sense-making (Adiredja, 2019; Adiredja et al., 2020). A researcher adopting this perspective:

> begins with the assumption that all students are capable of reasoning mathematically, and that they bring productive resources for learning mathematics. In research about student mathematical thinking, such perspective maintains flexibility with respect to the source and form of productive knowledge and reasoning. Important learning resources can stem from students' experiences from both in and out of the classroom, and productive sense-making can be expressed in imperfect mathematical language and with inconsistencies. In fact, inconsistencies and imperfections are sites for exploration for productive understanding. (Adiredja et al. 2020, p. 521)

Adiredja (2019) described a methodological framework for cognitive researchers who want to engage with anti-deficit work. The framework involved several criteria. First, researchers must engage in intentional selection of research subjects who are implicated in broader and individual deficit narratives. Second, researchers should adopt an anti-deficit cognitive theoretical framework, which allows them to construct an anti-deficit story using careful analysis of students' sense-making. Finally, researchers should explicitly challenge deficit interpretations of data.

We argue that teaching experiments as described by Steffe and Thompson (2000), which are grounded in a radical constructivism (von Glasersfeld, 1995), are well suited to support researchers in constructing an anti-deficit story via a careful analysis of students' sense-making. A foundational assumption of the teaching experiment methodology is that students' independently construct their own mathematical realities based on their repeated experiences making sense of their experiential world (von Glasersfeld, 1995); a researcher's goal is to use the student's words and actions to build models of their mathematical realities, with the resulting models referred to as the mathematics of students (Steffe & Thompson, 2000). Connecting teaching experiments to Noddings's (2002) care theory, Hackenberg (2010) identified that such a process involves cognitive decentering in which a researcher (or teacher) attempts to put aside their own reality and understand the mathematical reality of the student. Such a process "goes beyond just knowing that a student thinks differently to attempting to think like the student thinks, and acting upon that attempt to open possibilities for the student to make progress in some way" (Hackenberg, 2010., p. 240). Adopting this perspective, researchers using a teaching experiment methodology understand the mathematics of students as a form of legitimate mathematics, even when a student's mathematics may not align with researchers' or mathematicians' conceptions.

Adopting both a radical constructivist view and an anti-deficit perspective (Adiredja, 2019), there are no such things as misconceptions–only conceptions that have worked for students in their prior experience. Further, although a teacher's or researcher's task or prompt may support or occasion shifts in student meanings, they can never cause such shifts; instead shifts in students' meanings should always be attributed to the effortful sense-making on the part of the student (Adiredja, 2019; Steffe & Thompson, 2000). Finally, we note that both approaches de-emphasize formal mathematical knowledge as conceived by mathematicians. In fact, Steffe & Thompson (2000) go so far as to argue for mathematics of students becoming a foundation for school mathematics:

> By regarding mathematics as a living subject, we are faced with a different mathematics than appears in contemporary school mathematics… We strive to specify the mathematical concepts and operations of students and to make them the conceptual foundations of school mathematics. (p. 269)

Hence, we view radical constructivists' teaching experiments as a viable methodology researchers can use as they provide anti-deficit stories. We now describe how we used this methodology in ways that align with an anti-deficit perspective.

Methods, Participants, and Analysis

We describe a student's, Ariana's, sense-making during an exploratory teaching phase of a teaching experiment (Steffe & Thompson, 2000) in which we engaged her in the *Faucet Task*. This teaching experiment was part of a larger design-based research study in which the research team was interested in investigating the extent to which middle-school students could reason quantitatively and covariationally to conceive of and graphically represent relationships.[5] Although the research team was familiar with secondary and undergraduate students' reasoning in relevant contexts through their prior research, they had not yet investigated the ways middle-school students may engage in such reasoning and thus initially conducted exploratory teaching.

The goal in this exploratory teaching was "to become thoroughly acquainted, at an experiential level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest" (Steffe & Thompson, 2000, p. 274). For example, we were unsure the extent to which middle school students experienced opportunities to reason about and graphically represent relationships between covarying quantities. As such, we designed several tasks, including the *Faucet Task*, that allowed

[5] We refer the reader to Paoletti et al. (2020), Paoletti et al. (2022), and Paoletti et al. (2023) for additional findings from the larger design experiment.

us to explore the ways students may naturally reason about, and represent, such situations.

Due to the exploratory nature of this part of the study, our interactions with students were largely responsive and intuitive. During such interactions, a teacher-researcher's (TR's) actions are not pre-planned in advance of the session, instead relying on their in-the-moment conjectures about how and why students are reasoning during the interactions (Steffe & Thompson, 2000). Although exploratory teaching was largely our purpose in this teaching experiment (Steffe & Thompson, 2000), we audio and video-recorded each session with the intention of building viable models of the student's evolving mathematical meanings as we engaged her in a 10-session teaching experiment.

Subjects and Setting

The study occurred in a Northeastern U.S. school that hosts a diverse student population (over 75% students of color). We asked teachers to recommend students who could articulate their thinking and would be willing to participate. Particular to this paper, we characterize the activity of one Latina 6^{th}-grade student, Ariana, we engaged in the teaching experiment. We focus on the first three sessions in which Ariana addressed questions particular to the *Faucet Task*.

Consistent with the anti-deficit framework principle of intentional subject selection (Adiredja, 2019), we chose Ariana as according to her end of year state test, she was categorized as "Partially Met Expectations" (Level 2 out of 5). As our results will show, this score did not accurately capture Ariana's full mathematical capabilities. We highlight her brilliance as she engaged in sense-making to explore mathematical ideas well beyond what is typically expected of a 6^{th}-grade student.

Data Analysis

Consistent with the teaching experiment methodology (Steffe & Thompson, 2000), we used on-going and retrospective analyses to analyze the data. Both phases of analysis, involved conducting conceptual analysis—"building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 45). The conceptual analysis method allowed us to develop and refine models of Ariana's mathematics that viably explained her actions.

During on-going analysis, the research team met after each teaching episode to review the video and identify important instances in student activity that supported our building initial models of Ariana's mathematics to viably explain her observable words and actions. These initial models supported our designing and adapting tasks for future episodes. In these future episodes we tested these models by predicting how she might respond to a given task or situation. Such activity is consistent with analytical interactions in a teaching experiment (Steffe & Thompson, 2000).

During retrospective analysis, we again performed conceptual analysis (Thompson, 2008) to generate, test, and adjust models of Ariana's mathematics so these models provided viable explanations of her activity. The research team re-watched the entire teaching experiment sequentially to analyze the data using generative and convergent approaches (Clement, 2000). Using a generative approach, we watched videos identifying occurrences providing insights into Ariana's in-the-moment meanings (Thompson, 2016) for constructing, interpreting, and graphically representing relationships between quantities. Using these instances, we generated tentative models of her mathematics, including characterizing Ariana's situational epistemological justifications. Using a convergent approach, we tested these models by searching for supporting or contradicting instances in Ariana's other activities. When evidence contradicted our models, we revised our model and returned to prior data with these new hypotheses in mind. Further, we considered how researchers adopting a deficit view may describe Ariana's activity, and explicitly challenged such interpretations (Adiredja, 2019). This process resulted in a viable model of Ariana's mathematics that allows us to tell an anti-deficit story about her mathematical learning.

Developing Epistemological Justifications to Reason Emergently: The Case of Ariana

We describe Ariana's activity addressing the *Faucet Task*, first highlighting her experiencing a situational intellectual need she resolved by constructing a situational epistemological justification via a quantitative structure. We show how she leveraged this justification as she described how various mathematical objects varied. We conclude by highlighting how this activity supported Ariana in addressing questions regarding 'function' (from the researchers' perspectives) but illustrate that these questions did not stem from or produce an intellectual need for her.

Developing a Situational Epistemological Justification in the Faucet Task

When first presented with the *Faucet Task* and asked to "play around" with the knobs, Ariana identified "how much water comes out" and "temperatures of the water" as quantities she could measure. As Ariana addressed Scenarios (A)–(D), she constructed and re-constructed particular quantitative and covariational schemes and operations to make sense of the situation (e.g., reasoning about directional changes in two quantities' magnitudes, making additive comparisons to describe more water leaving the faucet). This activity formed a basis for her developing situational epistemological justification. For instance, addressing Scenario (A), Ariana drew on her personal experiences with faucets to accurately describe "there's going to be more water coming out," and when asked what was going to happen to the temperature, she said, "it's going to become colder."

When addressing Scenario (B), Ariana initially indicated both water temperature and amount of water would decrease. However, when asked to justify the change in temperature she re-considered:

TR: Why do you think it's going to get colder?

A: Because you're turning it off [*pauses*].

TR: We're turning cold off, so.

A: [*interjecting*] It would become warmer.
TR: Why would it become warmer?
A: Because, since we're turning it [the cold knob] off. Um the more you turn it, um to the right [*referring to Scenario A*], the more colder it would get. But since we're turning it um to the left, it would become warmer because we're basically turning it [the cold knob] off.

Addressing the TR's prompt, Ariana reconsidered the quantities in the situation, arguing that whereas turning cold water on resulted in a decrease in temperature in Scenario (A), turning the cold off would result in an increase in temperature in Scenario (B).

Leveraging situational intellectual need and situational epistemological justification, we contend Ariana experienced a situational intellectual need as she attempted to justify her initial conjecture that water temperature decreased for Situation (B). She experienced a minor perturbation when she attempted to use her already existing schemes and operations to make sense of and mathematize a novel situation. Reconciling this need, Ariana used existing schemes and operations (e.g., reasoning about the directional change of quantities) to begin to construct a situational epistemological justification that enabled her to determine how, and more importantly why, the quantities in the situation varied as they did (i.e., arguing that since turning the cold water on results in colder water, turning it off will result in warmer water). Specifically, her constructed quantitative structure entailed schemes involving compensation such that she could anticipate changes in temperature regardless which knob was changed. This is reflected by the fact that Ariana had no difficulty in accurately predicting how each quantity would change for Scenarios (C) and (D).

We note that a researcher adopting a deficit perspective may consider Ariana's sense-making insignificant given the everyday context. However, we counter such an interpretation by highlighting the sophistication of this reasoning. Particularly, Ariana understood that a modification to one knob can cause a temperature change more directly related to the other knob (e.g., "[the temperature] would become warmer because we're basically turning [the cold knob] off"). Such reasoning requires a situational epistemological justification that entails a complex relationship between (at least) three interrelated quantities (amount of hot knob turns, amount of cold knob turns, and water temperature).

Leveraging a Situational Epistemological Justification Addressing Graphing Prompts in the Faucet Task

Ariana leveraged her developing situational epistemological justification when describing how the mathematical objects (seen in Figure 3a/b) varied in the next two applets. For example, after describing the point as moving according to the endpoints of the two varying segments (Figure 3b), Ariana predicted and then tested how the point moves for Scenarios (A)–(D). In each case, Ariana leveraged her situational epistemological justification to accurately address each prompt. In one instance, addressing Scenario (D), Ariana first described that the temperature and amount of water decreased and that this corresponded to each segment decreasing in length. She then described, "since they're [*motioning to the segments on the axes*] both moving, it's [the point's] going to go diagonally [*motioning from the point on the computer screen diagonally down and to the left*]." In each scenario, Ariana described that the point's movement was dictated by the covarying magnitudes, which she later built on to describe the direction of the emergent trace in these scenarios.

Ariana's activity on the last part of the *Faucet Task* (e.g., Figure 4**)** provided an opportunity to explore if she was engaging in emergent graphical shape thinking. Addressing the first graph (Figure 4a), and indicative of reasoning emergently, Ariana experienced a minor perturbation as she immediately questioned if the graph "started down here [*pointing to the bottom right endpoint*] or up here [*pointing to the top left endpoint*]?" She then argued if the graph started at the top left endpoint, then "turning cold on" would produce the given graph. Justifying this, she put her finger over the top left endpoint and indicated for the initial state, "If it started here, the hot water, the hot water would be on." Then, leveraging her situational epistemological justification, she argued the action that would result in the given graph was "turning cold on… because if you turn cold on it [water temperature] would go down [*motioning along the curve from the top left endpoint*] and as you can see it's a little curve [*motioning over the curve near the bottom right endpoint*] as if the water is increasing [*motioning horizontally along the horizontal axis to indicate the amount of water is increasing*]." Shortly thereafter, Ariana argued if the graph started at the bottom right endpoint, then "turning the cold water off" would produce the graph traced in the opposite direction.

We infer Ariana experienced a situational intellectual need as she was tasked with describing a single knob turn that would produce the given graph but was unsure which direction the graph traced (e.g., questioning the starting point of the graph). Ariana resolved this perturbation by using her existing situational epistemological justification in a new way. In particular, she interpreted one graph in two different ways and provided two different descriptions of starting states and turns that would accurately produce the graph. Hence, we infer Ariana was engaging in emergent shape thinking. Ariana's emergent reasoning is particularly powerful as there is evidence that such reasoning is non-trivial for pre-service and in-service mathematics teachers (Moore et al. 2019a, Thompson et al., 2017).

Ariana Explicitly Addressing Questions about Univalence

Consistent with exploratory teaching, the TR next opted to explore in-the-moment conjectures. In particular, he conjectured that Ariana's quantitative structure could support her in considering scenarios that were more complex than the applet was designed to address. He intended to explore if Ariana, similar to Arya in Paoletti and Moore (2017), could conceive of and describe both functional and non-functional relationships (from his perspective) within the scenario.

First, the TR prompted Ariana to imagine if she could turn both knobs simultaneously, which was not possible in the applet as designed. He intended to explore if Ariana might consider novel situations that may produce different changes in the temperature and amount of water than she had yet experienced. He then asked if she could describe "a way to turn both of them to keep the temperature the same." The following conversation ensued:

> A: If you turn them both on, the temperature would stay the same… if it started off like equal, and you left it [the amount you're turning each knob] equal, but you still move it, it will, the temperature will stay the same.
>
> [*Ariana spontaneously considered what happens to volume as she turns one knob on and the other off. The TR returned to keeping temperature invariant before moving to the situation Ariana was describing*].
>
> TR: But the volume would in that case, what would happen to the volume if turning them both on by the same amount?
>
> A: The volume would increase.
>
> TR: Increase. What if we were turning them both off by the same amount?
>
> A: Um it [volume] would decrease.

Despite the applet not allowing Ariana to turn both knobs simultaneously, she was able to make a modification to her quantitative structure by imagining a new situation that entailed simultaneously turning the knobs in the same direction. She reasoned in such a case water temperature remained constant while the amount of water varied. We infer Ariana was implicitly reasoning that situationally the amount of water is not in a univalent relationship with temperature (i.e., the same temperature magnitude can correspond to multiple amount of water magnitudes).

Immediately after this, the TR began to question whether every amount of water magnitude corresponded to exactly one temperature magnitude:

TR: I don't want any more water…So we want that same amount of water. But we want it to be hotter, and you can turn both knobs.

A: But you would just turn on the hot.

TR: If I turn on the hot more, it's going to increase the temperature and the amount total amount of water right? So say I turn the hot on a little bit. But at the same time I turned the cold off a little bit. What would happen in that case?

A: In that case, the volume would stay the same, because you're adding a piece that you already took away from the cold.

TR: Right… And what happens to the temperature?

A: Um the temperature increases.

TR: Increases. What if I want the water to be a little colder?

A: You would take, you [do] the opposite, you would just take away the hotness, you take, like a turn of the hotness [off] and then add another turn of cold.

Leveraging her situational epistemological justification, Ariana understood if she simultaneously turned the knobs in opposite directions by equal amounts, the amount of water leaving the faucet would remain invariant, but the water temperature would vary. We infer Ariana implicitly reasoned that situationally temperature is not in a univalent relationship with water volume (i.e., a particular volume magnitude can correspond to several temperature magnitudes).

As Ariana's quantitative structure supported her in making decisions regarding univalence in each case, the TR conjectured she may be able to characterize whether even more complex relationships were univalent. Hence, he prompted Ariana to consider if each coordinate point representing *(Amount of water, Temperature)* magnitudes corresponded to exactly one situational state, e.g., one pair of *(Hot knob turns, Cold knob* *turns)*. Referring to a specific *(Amount of water, Temperature)* coordinate point shown on the applet, he asked Ariana if it was possible to play with the knobs to obtain both the same amount of water coming out and the same temperature. After a 6-second pause, Ariana indicated this was possible. The following conversation occurred:

A: I'm not sure that the temperature, but… if we just like add, umm, another piece of hot water and take a cold water away, umm, [the pink segment on the horizontal axis, seen in Figure 2] would stay the same… because the amount of water is coming out.

TR: Yeah, so the pink will stay the same, but do you think it [the point] would move up or down or would it stay there?

A: It would move up because you're, more hot water. Now that you told me that, I don't think, I don't think there's a way.

Ariana initially additively coordinated the volume of hot water and the volume of cold water to consider how to maintain a constant total volume of water. When asked if the point would move up or down, Ariana turned her attention to temperature, realizing it would increase in the situation she described.

After this, the TR provided Ariana with another point on the graph and asked if another mixture of hot and cold water could produce that same *(Amount of water, Temperature)* coordinate point. After an eleven second pause, and consistent with her initial response above, Ariana attended only to the amount of water to conclude another situation could produce the point. Also consistent with her response above, when testing her conjecture Ariana then attended to temperature, realizing the temperature changed and that her proposed situation produced a different point. After Ariana realized this, the TR asked "So if I stopped [at] a specific place…I have that amount of water coming out and that temperature [*pointing to the segments on the axes respectively*]. Any other situation gets me there?" Ariana immediately responded that this was not possible just prior to the session ending. Due to time constraints of the session, the TR did not have an opportunity to further explore Ariana's meanings for this complex relationship.

A researcher adopting a deficit account may characterize Ariana's activity above as showing a lack of sense-making; in both cases she first only attended to one quantity while considering the TR's prompt. We challenge such an interpretation by highlighting the complexity of considering four changing quantities simultaneously (i.e., hot knob turns, cold knob turns, temperature, and amount of water). In each case, Ariana successfully coordinated three changing quantities prior to considering the fourth; such multivariational reasoning is non-trivial for students from middle school to advanced mathematics (Jones, 2022; Panorkou & Germia, 2020).

Reflecting on Ariana's actions, we infer that she continued to (re)construct her quantitative structure and, thus, her associated situational epistemological justification involving relationships between states of the turning knobs and the resulting temperature and amount of water. She concluded in the moment that two different knob states could not produce the same *(Amount of water, Temperature)* coordinate point. That is, Ariana concluded that the relationship between *(Hot knob turns, Cold knob turns)* states and *(Amount of water, Temperature)* was a univalent relationship. Hence, we infer Ariana described three relationships as having or not having the property that "every value of one quantity determines exactly one value of the other [quantity]" (Thompson & Carlson, 2017, p. 444; Table 1).

Table 1

The Relationships Ariana's Considered as Possibly Representing Covariational Functions

Situation	"One quantity"	"The other [quantity]."	Univalent?
Turning both knobs in same direction	Temperature	Amount of water	No
Turning both knobs in opposite directions	Amount of water	Temperature	No
Turning either knob any amount	*(Temperature, Amount of water)*	*(Hot knob turns, Cold knob turns)*	Yes

Ariana developed situational epistemological justifications that supported her in determining if certain quantitative relationships had the property of univalence. However, univalence was a natural (although not always conscious) aspect of her quantitative structures; Ariana never experienced a perturbation regarding if a relationship was not (or was) univalent. Hence, Ariana did not experience any intellectual need for explicitly considering the possible univalence of the relationships. Ariana only determined if a relationship was univalent because the TR prompted her to do so. Rabin et al. (2013) referred to such a situation as entailing a social need, rather than intellectual need, and noted "for students to learn the mathematics we intend to teach them, they must see a need for it, where 'need' means intellectual need, not social or cultural need" (p. 652). In such situations, we agree with Harel (2008) who argued students are less likely to learn what teachers or researchers intend when "students' actions are socially rather than intellectually driven" (p. 488).

Univalence, Intellectual need, and Function

Like Ariana, there is little evidence most students (or teachers) experience any type of intellectual need that motivates constructing an epistemological justification for univalence, a property critical to a set-theoretic definition of function. For example, in her study of prospective secondary mathematics teachers, Even (1990) noted, "Some serious questions are raised by the fact that, without prompting, none of the subjects could come up with a reasonable explanation for the need for the property of univalence" (p 531). Compatible with Harel's (2018a) description of presenting mathematics that makes students feel like aliens in their knowledge construction, Even (1990) characterized current approaches to the teaching of function as contributing "to making mathematics look like an arbitrary collection of rules and definitions" (p. 531).

Reflecting on the collective body of research on students' understandings of a set-theoretic function definition, Even and Bruckheimer (1998) questioned emphasizing univalence for pedagogical purposes, instead suggesting researchers and educators consider the historical development of function including its initial roots in relationships between variables. The covariational meaning of function characterized by Thompson and Carlson (2017), described in the opening of this paper, and as exemplified in Ariana's thinking, fits this suggestion. Rather than foregrounding univalence, Thompson and Carlson's (2017) covariational meaning emphasizes a student initially *constructing* invariant relationship(s) between quantities. Once a student has constructed such a relationship (and potentially a complex network of relationships), she can investigate properties of the relationship(s). Univalence is one possible property of a relationship (or a property common across a network of relationships; see Table 1). As Ariana's example illustrates, a student can construct an invariant relationship situationally and consider certain properties of that relationship, without concerning herself with formal mathematical representations such as graphs or algebraic rules (Paoletti & Moore, 2017, 2018; Thompson, 2011).

Concluding Remarks

In this paper, we extended Harel's (2008, 2018a, 2018b) constructs by defining situational intellectual need and situational epistemological justification in the context of constructing quantitative structures. We then presented an anti-deficit story (Adiredja, 2019) exemplifying Ariana's powerful sense-making as she experienced situational intellectual needs, which she resolved via the creation of a situational epistemological justification.

Like Arya, the undergraduate student in Paoletti and Moore (2018), Ariana's quantitative structure supported her in reasoning emergently to (re)construct and interpret graphs as representing "simultaneously what is made (a trace) and how it is made (covariation)" (Moore & Thompson, 2015, p. 785). Further, her case highlights the extent to which a student can (re)construct a situational epistemological justification that entails a quantitative structure to consider various relationships in and properties of this structure, regardless if these relationships maintain the property of univalence.

We are not surprised students may not be spontaneously motivated to determine whether a relationship is univalent when asked to mathematize a novel situation. Univalence is unlikely to be critical to their reasoning as it is merely a byproduct of their constructing quantitative structures. Ariana, and possibly the pre-service teachers in Even's (1990) study, had not yet experienced any intellectual need for the property of univalence. Consistent with Harel's (2008, 2018a, 2018b) arguments, we contend it is unlikely for students (or teachers) to appreciate the importance of univalence until they have experienced some intellectual need for it, and it is only then that they will come to value the property of something we, as mathematicians or mathematics educators, refer to as 'function'.

We question current approaches to teaching a function definition early in students' school experiences (e.g., Ayalon & Wilkie, 2019), and then focusing almost solely on functional relationships in secondary school. In fact, we conjecture this approach makes certain topics more complicated than if we allowed for non-functional relationships. For instance, Paoletti et al. (2015) found that most pre-service teachers used procedures when asked to graph the inverse of trigonometric function that was distinct from the procedures they used for non-trigonometric functions. However, this is unsurprising given the time and attention standard approaches to teaching inverse trigonometric functions dedicate to students memorizing various domain and range restrictions for different trigonometric functions. We conjecture an approach focusing on supporting students in developing meanings for trigonometric functions and their inverses as representing the same underlying relationship, regardless of function-ness, as in Paoletti (2020), would be more productive.

Collectively, we believe current approaches to function in school mathematics are likely over-privileging the use of formal mathematical definitions, which can "insidiously de-value students' informal mathematical knowledge and emerging understandings" (Adiredja & Louie, 2020, p. 43). Returning to the horse and cart analogy, Ariana's activity exemplifies reasoning covariationally can provide a younger student the horse needed to pull the cart that is properties critical to a formal set-theoretic function definition. However, her covariational reasoning did not lead to Ariana experiencing an intellectual need for the cart itself. Although other researchers may view this as a deficit in Ariana's reasoning, we argue the anti-deficit story illustrates what was important to Ariana's sense-making (constructing relationships between quantities) and what was not significant (set-theoretic properties of function). As such, and contrary to the suggestions of others (cf. Ayalon & Wilkie, 2019), we propose introducing a set-theoretic function definition to students only after they have experienced an intellectual need for it and its properties (e.g., exploring the analysis of relationships in the context of concepts like differentiation and integration). We suggest freeing the horse from the cart as the horse can do the same work for the student with, or without, the cart. Just as we do not introduce the definition of "polygon" prior to introducing students to triangles and rectangles, nor the definition of "group" prior to introducing students to the lines of symmetry of a square, we question the value of introducing a set-theoretic definition of function early in students' mathematical experiences that can serve to make students feel like "aliens in knowledge construction" (Harel, 2018a, p. 38).

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