The Mathematics Educator
2024 Vol. 32, No. 1, 158-187

Professional Noticing of Student Thinking in the Context of Mathematical Modeling

Rui Kang

This study examines how professional noticing of student mathematical thinking evolves in the context of modeling and what is special about the context of modeling. Eight middle and secondary school teachers participated in this one-semester long study and received three training sessions on professional noticing. Teachers provided more substantial detail about the mathematical aspects of student strategies after participating in the training, and they benefited from conversations with colleagues scaffolded by three phases of professional noticing: attending to, interpreting, and responding to student thinking. Teachers also identified specific aspects of student thinking in the context of modeling, especially local conceptual development.

The mathematical modeling process tends to produce rich discourse and abundant artifacts (Lesh & Doerr, 2003). Therefore, it can be used to develop what Jacobs et al. (2010) called "professional noticing of children's mathematical thinking" (p.170). However, this potential of mathematical modeling has not been fully explored. One exception is a study by Alwast and Vorhölter (2022), which found that teachers' noticing of student thinking in the context of modeling is important for helping students solve modeling tasks because students often do not directly express their difficulties and instead rely on their teachers to notice and make sense of their strategies.

Background

I first introduce the definition of modeling, followed by a presentation of the theoretical framework of this study and an

***Rui Kang** is a Professor in Secondary Education at Georgia College & State University. Her primary research focuses on teacher preparation and professional development in mathematics and science education.*

explanation of why mathematical modeling provides a special context for developing noticing expertise.

Defining Mathematical Modeling

Mathematical modeling is a process of developing systems of mathematical objects (e.g., operations, relationships, rules) that can be used to describe, explain, and predict a wide variety of phenomena in natural sciences (e.g., physics, chemistry, biology), engineering, technology, and social sciences (e.g., history, politics/policies; Doerr & English, 2003). It is a cyclic and iterative process that starts with a realistic situation and ends with a reporting of mathematical results. Mathematical modeling is complex and often involves the following steps. First, a realistic situation is read, understood, simplified, and systematized. Second, the situation is mathematized and represented symbolically. Third, an emerging model is generated and then subjected to a validity test against the situation. Fourth, a decision is made about the validity and limitations of the emerging model. Finally, the process is revisited as the emerging model is revised and refined. The product of modeling usually is not a definitive solution but multiple plausible solutions and sometimes conceptual tools and artifacts (Galbraith & Stillman, 2006).

There is a critical difference between the traditional perspective on solving word problems and mathematical modeling. The traditional way of solving a word problem can be viewed as a transition from mathematics to reality or *realizing mathematics,* whereas mathematical modeling is a transition from reality to mathematics or *mathematization* (Wess et al., 2021). Traditional word problems, in which the situation has already been mathematized for students, sometimes are referred to as “pre-modeled problems,” “illustrative applications,” or “dressed up” word problems (Blum & Niss, 1991), but mathematical modeling tasks require students to develop symbolic descriptions of realistic problem situations. Another major difference between “dressed up” traditional word problems and mathematical modeling is that the validation of real results is necessary. The traditional perspective on solving word problems does not require students to go through the modeling cycle.

Researchers of mathematical modeling (e.g., Cetinkaya et al., 2016; Doerr, 2007) have noted that for teachers to effectively enact mathematical modeling in their classrooms, they need a broad and deep understanding of various solution methods including unexpected ideas from their students. This is a highly cognitively demanding expectation of teachers and "requires rapidly seeing, understanding, and interpreting … multiple ways of thinking" (Cetinkaya et al., 2016). In other words, successful enactment of mathematical modeling in the classrooms depends on a teacher's ability to notice and make sense of student thinking (Jacobs et al., 2010). This means that on one hand, developing noticing skills will enhance teachers' abilities to successfully teach modeling; on the other hand, mathematical modeling as a complex and often open-ended, ambiguous process (Lesh & Doerr, 2003) provides abundant opportunities to further develop teachers' noticing of student thinking.

Theoretical Framework: Professional Noticing of Children's Mathematical Thinking

Goodwin (1994) defined *professional vision* as "socially organized ways of seeing and understanding events" (p. 606) that enables members of a profession to view and make sense of complex situations from a unique perspective. Similarly, Stevens and Hall's (1998) notion of *disciplined perception* and Mason's (2002) *intentional noticing* have also been used to distinguish perceiving or noticing in professional settings from similar activities in everyday life. Inspired by these various concepts, Jacobs et al. (2010) developed the *professional noticing of children's mathematical thinking framework* to explore how mathematics teachers view and make sense of their complex classroom environments from a professional lens. Instead of noticing every single aspect of teaching and learning, Jacobs et al.'s framework has a particular focus on children's mathematical thinking and include three distinct but interconnected components of teacher noticing: "attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings" (p. 172). More specifically, *attending to children's strategies* is to identify noteworthy thoughts and important mathematical details in children's strategies, which requires teachers to use their expertise to discern patterns and chunk information in complex instructional situations. *Interpreting children's understandings* is to provide productive, evidence-based explanations of children's strategies. Interpreting is deeper than attending to and requires analysis and reasoning supported with evidence. *Deciding how to respond* is to propose next steps or next problems based on what teachers have learned about children's understandings. It is a type of intended responding and requires no execution of the response (Jacobs et al., 2010).

Noticing students' ways of thinking is an essential professional skill for teachers, and therefore, mathematics educators need to search for ways to help teachers develop this skill within a variety of contexts. Research has shown that noticing is teachable and can be developed through conversations with teachers (Sherin & Van Es, 2009). For example, Sherin and Van Es found that teachers in their study initially tended to focus on all the parties in the video they were discussing, including the teacher, students, and sometimes an administrator. In addition, their "general approach for making sense of the issue under discussion" (p.24) was either descriptive or evaluative. However, after a year-long professional development, the teachers were able to focus their attention on students' mathematical thinking. They were also able to refrain from premature judgement and instead take an interpretive approach to understand students' thinking. Noticing can be developed in a variety of specific mathematical domains such as the derivative concept, early numeracy, and algebraic thinking (see Sherin et al., 2011).

Connecting Modeling and Noticing

Modeling provides a carefully controlled and mathematically enriched environment that serves as an ideal context for supporting the development of noticing. *Model*

Eliciting Activities (MEAs) are *thought-revealing problems* that challenge students to develop constructs (e.g., a concept system or a model) to represent, interpret, and provide solutions for open-ended mathematical situations (Lesh & Doerr, 2003). Lesh and colleagues (e.g., Lesh & Doerr, 2003; Lesh & Harel, 2003) observed that when their students were engaged in MEAs, they were able to improve or modify their initial interpretations or conceptualizations of the underlying constructs of a problem situation within a relatively short period of time (typically between 60-90 minutes), a phenomenon these researchers referred to as *local conceptual development*. This phenomenon bears striking similarities to the stages of development outlined by Piaget (1950), albeit the development is situated within one task or a small set of similar tasks. The word "local" is used to signal the rapid conceptual development observed in the modeling process in contrast with the stages of human cognitive development typically observed over several years.

The process of local conceptual development not only cultivates mathematical habits of mind (e.g., revision and refinement of ideas, perseverance, productive struggles) but also makes student thinking visible and noticeable. When initial mathematical models are evaluated against each other or against reality, between-model mismatches and model-reality mismatches often lead to the development and refinement of conceptual tools (Lesh & Doerr, 2003). Between-model mismatches often occur when several competing models are proposed in a group, and these models reflect alternative ways of thinking. This is expected in the context of modeling since there is often more than one way to solve a modeling problem or multiple solutions can be equally valid. Model-reality mismatches often occur when a model is found to be inconsistent with a real-life situation or requirements. These mismatches often lead to cognitive dissonance as well as productive struggles, which prompt the learner to revise their initial way of thinking to resolve the mismatches (Lesh & Harel, 2003). When teachers notice local conceptual development, they often feel excited and sometimes surprised. This helps teachers take on an asset-based perspective of their students' abilities.

The purpose of this study is to identify special focuses and trajectories for professional noticing of student mathematical thinking in the context of modeling. Two research questions guide this study: (a) How is noticing developed in the context of modeling? (b) What is special about the context of modeling for developing noticing?

Methods

Participants and the Context

Participants included eight mathematics teachers enrolled in a secondary (6-12) teacher preparation program at a mid-sized public university located in the southeastern region of the United States. All the teachers were non-traditional students seeking a new career, and their ages ranged from mid-20s to late-40s. Six of the eight teachers were high school mathematics teachers; two were middle school mathematics teachers. These teachers taught a variety of mathematics subjects at the time of the study, including algebra, geometry, statistics and probability. Their years of teaching experience ranged from 0 to 5 years. One of the participants in my study was receiving training to become a teacher and was completing a master's degree in education; the rest were practicing teachers who were beginning, full-time classroom teachers on provisional certifications. None of the teachers reported taking a course in mathematical modeling or engaging in any formal training on how to teach mathematical modeling before this study. Although they participated in a professional noticing activity in an earlier mathematics methods course, none of them had any prior experience of applying Jacobs et al.'s (2010) framework in the context of modeling. More information about the teachers and students, along with the contexts of this study, is provided in Table 1.

This entire study was conducted in a 15-week, semester-long, student teaching/internship course that was taught during the COVID global pandemic. I (the researcher) was the course instructor and ran all the sessions of the study. Prior to professional noticing, I engaged the teachers in the following

Table 1
Background Information for the Videos

Group & Video #	Teachers*	Task & Main Underlying Ideas or Concepts	Teacher Who Enacted the Task	Students*
1	Alicia, Eric, Carrie Katherine	*COVID-19 – Where would we be now??* (see Appendix A) Exponential functions, curve fitting, multiplicative reasoning	Katherine	Josh, Lily, Olivia 9th graders
2	Alexis, Ann, Brian, Sarah	*We're Moving!* (see Appendix B) Ranking, aggregating ranked quantities, weighting ranks, and transforming data sets	Brian	Emma, Jack, and Liam 7th graders

Note. *All names are pseudonyms.

activities to help them develop theoretical knowledge about modeling. First, following the recommendation of previous researchers (e.g., Anhalt & Cortez, 2016), I asked the teachers to work through several MEAs in small groups and then to reflect on their group's modeling process and analyze the characteristics of the MEA tasks.

Second, the teachers collaboratively designed modeling tasks in their small groups. Group 1 created a task that asked students to generate a mathematics model using their knowledge about functions to predict the number of COVID cases based on data over a period of 109 days and then relate graphs of COVID cases to the mitigation policies adopted by various states (see Appendix A). Group 2 chose to adapt an MEA from Doerr and English (2003) that they themselves had worked on earlier in the course. They changed the context of the original task to help clients decide which state to move to during the COVID-19 pandemic using ranking and weighting (see Appendix B).

Two teacher volunteers, Katherine and Brian, each worked with a small group of three students and recorded their instruction via Zoom due to the Institution Review Board (IRB) COVID-19 pandemic guidelines and regulations. Katherine taught the first task, which involved exponential functions, curve fitting, and multiplicative reasoning. Two of Katherine's students, Josh and Lily, were studying exponential functions at their school when the task was implemented, whereas Olivia had not yet learned about exponential functions. Brian taught the second task, which involved ranking, aggregating ranked quantities, weighting ranks and were not included in the middle school mathematics curriculum. Brian's three students had no formal exposure to these concepts at their school. Neither task required students to collect additional data. Although students were expected to use their prior knowledge to solve these tasks, they were not directed to use any specific concepts or methods.

I divided the professional noticing training into three two-hour sessions so that teachers had the opportunity to interact with the same video multiple times (Han et al., 2023). More specifically, teachers in Group 1 watched Video 1 in which Katherine enacted Group 1's modeling task with her three students. Similarly, Group 2 watched Video 2 in which Brian enacted Group 2's modeling task with his three students. Jacobs et al. (2010) recommended that teacher educators use discussion prompts to support teacher candidates and practicing teachers to develop professional noticing expertise. These researchers provided a couple of broad prompts for each of the three phases of their framework. I expanded Jacob et al.'s list of discussion prompts based on the context of this research (Table 2).

The activities in each of the three sessions are outlined below:

- In the first professional noticing session, I instructed the teachers to independently take in-the-moment notes of what they observed on the video using the discussion prompts in Table 2 as a guide.
- In the second professional noticing session, I engaged the teachers in video clubs (Sherin & Van Es, 2009) facilitated through Jacobs et al.'s (2010) professional

noticing framework. Besides the video, each group of teachers also analyzed the students' written responses and triangulated the students' strategies observed on the video with the strategies in the written responses.

- In the third and last professional noticing session, teachers watched their group's video again individually and answered the same prompts in writing. After the last session, I also conducted a semi-structured interview with each teacher and asked them to explain to me the written responses they wrote in the third session.

More details are warranted for the second session. Video clubs are "video-based professional development environment" (Sherin & Van Es, 2009, p. 20), in which teachers learn how to teach by observing and analyzing student learning, and researchers/educators study teachers' learning simultaneously. I selected video clips (between 1 and 10 minutes) for each group of teachers from the videos (i.e., Video 1 and Video 2), which featured critical incidents (Tripp, 2011) or mathematically rich episodes during which student thinking were the most visible. Ulusoy & Çakıroğlu (2020) found that raw classroom videos include "useful and useless events which reduce the productivity of group discussions in teacher education" (p. 259). However, micro-case video clips not exceeding ten minutes and featuring noteworthy instances were the most likely to generate productive conversations about student thinking. Teachers watched the entire video but focused on the short clips with playback and stops upon request.

Data Analysis

I recorded all the interviews and professional development sessions using the Gallery View with Shared Screen setting in Zoom to ensure both participants and the written artifacts were visible in the recordings. Transcripts of semi-structured individual interviews, student written artifacts, and the teachers' written responses constitute the data corpus of this study. I entered both interview transcripts and written documents/artifacts into the ATLAS.ti.9 data management

Table 2

Scaffolding Prompts for Professional Noticing

Phases	Prompts
Attending	1.* Please describe in detail what you think the group did in response to this task. 2.* Please describe in detail what you think each student in the group did in response to the task.
Interpreting	1.* Please explain what you learned about these students' understandings as a group. 2.* Please explain what you learned about each of the students' understandings. 3. Please explain what you think are the strengths in these students' understandings either as a group or as individuals or both. 4.Please explain what you think are the weaknesses in these students' understandings either as a group or individuals or both. 5.Please explain what you think are the major obstacles in students' understanding either as a group or individuals or both. 6.Did any students' response surprise you? Why or why not? 7. How do you describe students' process of solving the model task? How is it similar to or different than your own approach to the same task?
Deciding how to respond	1.* Pretend that you are the teacher of these students. What task or tasks might you propose next? 2.What other pedagogical strategies come to your mind? In other words, what comes next in your instruction? 3. What would you do similarly as the instructor in the video, why? What would you do differently and why? 4.* Overall, did the task turn out as you had imagined? Why or why not?

Note. * Questions included in the original prompts in Jacobs et al. (2010).

program and read and re-read the data before the beginning of the coding process. I used a hybrid approach of deductive (theoretically guided, derived from prior research) coding and inductive (emergent) coding to analyze the entire data (Fereday & Muir-Cochrane, 2006). I coded *attending*, *interpreting*, and *deciding on how to respond* separately. For *attending*, my analysis was initially informed by coding categories derived from the literature, but codes that emerged during the analysis were also allowed to avoid premature closure. Van Es and Sherin (2008) identified four "topics" that teachers typically notice in a busy classroom: *mathematical thinking*, *pedagogy*, *climate*, and *management*. These four topics served as my initial codes, but I used them as sensitizing concepts for "seeing, organizing, and understanding experience" (Charmaz, 2003, p. 259) and did not force or impose them onto my data (Patton, 2002). One of them survived data analysis, *mathematical thinking*. One code was discarded due to lack of match in the data. Specifically, management issues were not observed in the videos since each teacher only worked with a small group of three students. Two codes were modified to better fit the context of this study. *Climate* was changed to *the Zoom Environment* because the entire study was conducted during the COVID global pandemic via Zoom. *Pedagogy* was changed to *teachers' role as a facilitator* because the only pedagogical aspect that caught the teachers' attention was their own or their colleague's role as a facilitator. An additional code, *group dynamics and student interactions*, emerged from the data.

In addition, I further divided the code, *mathematical thinking*, into several subcategories to capture the special characteristics of mathematics thinking in the context of modeling. Earlier I mentioned that I was interested in whether the teachers would be able to recognize the students' *local conceptual development* as related to the task they were solving (Lesh & Harel, 2003). Prior research also emphasized the importance of identifying *blockages* (e.g., Galbraith & Stillman, 2006), *model-reality mismatches* (e.g., Lesh & Doerr, 2003), and *multiple solutions or between-model mismatches* (e.g., Borromeo Ferri, 2006; Lesh & Doerr, 2003). These four codes served as my starting point for coding mathematical thinking. I was also open to emergent codes, and identified a new code, *breakthroughs*.

For *interpreting*, I also started with codes found in prior research. In particular, Jacobs et al. (2010) emphasized the difference between (a) *making sense of* students' strategies grounded in evidence and (b) making *snap evaluations* based on minimal evidence. Therefore, my initial codes for interpreting were *snap evaluation* and *sense making*. As analysis went on, I found *sense making* was too broad to capture the nuanced interpretations made by the teachers. I then turn to open coding (Strauss & Corbin, 1990) and found two emergent codes: *mismatches driving local conceptual development* and *inflexible thinking due to rule-based pedagogy*.

Finally, prior research provided limited guidance for analyzing and *deciding on how to respond*. Therefore, I relied on open coding to categorize teachers' anticipation of their future strategies and identified two categories of strategies: *behavior-based strategies* and *thinking-based strategies*. The final coding scheme along with a definition of each code is displayed in Table 3.

For establishing the validity of data analysis, I applied multiple rounds of coding to the data and used the constant comparative method (Strauss & Corbin, 1990) iteratively to refine my initial codes until I reached a well-justified, robust data-code match. In addition, I used the constant member checking technique during the interviews, for example, rephrasing a response from a teacher and then asking: "Am I understanding your thoughts correctly?" I also had the opportunity to share my initial drafts of research findings with the teachers during a follow-up study and invited them to point out any inaccurate portrayal of their views. Teachers read my initial draft but provided no further suggestions. Finally, to improve the reliability of the study, I shared all the relevant data excerpts with an expert in mathematics education who served as a "critical friend." We coded the data excerpts separately and resolved our discrepancies collaboratively through peer debriefing.

Results

Below, I contrast the teachers' initial, in-the-moment notes during the first session of this study with their verbal and written responses during group discussions and semi-structured individual interviews to show shifts in each of the three phases of professional noticing: attending, interpreting, and deciding on how to respond.

Table 3
The Final Coding Scheme

Codes	Definitions
Attending	
The Zoom Environment	Instruction and interactions in a Zoom setting
Group Dynamics and Student Interactions	The roles students play in their group and how well they work with each other
Teacher's Role as a Facilitator	A teacher's behavior and verbiage when performing the facilitator role during enactment of modeling
Mathematical Thinking—Breakthroughs	An important revelation or discovery that leads to a successful solution
Mathematical Thinking—Blockages	Progress cannot be made after a significant amount of time (> 10 minutes) has lapsed
Mathematical Thinking--Model-reality Mismatches	A model is found to be inconsistent with a real-life situation or a client's requirements
Mathematical Thinking—Multiple Solutions or Between-model Mismatches	Different ways or paths to solve a problem or several competing models demonstrate alternative ways of thinking.
Mathematics Thinking—Local Conceptual Development	Improvement or modification of initial interpretations or conceptualizations of a problem situation within a relative short period of time
Interpreting	
Snap Evaluation	Judge students' strategies as good or bad without citing evidence from students' work or verbiage
Mismatches Driving Local Conceptual Development	Students demonstrated local conceptual development due to the necessity to resolve mismatches
Inflexible Thinking due to Rule-based Pedagogy	Students demonstrate inflexibility in solving a problem encountered during modeling that can be explained by the rule-based pedagogy they are used to
Deciding on How to Respond	
Thinking-based Strategies	Proposed strategies rooted in student thinking observed during the attending and interpreting phases of professional noticing
Behavior-based Strategies	Proposed strategies rooted in observations of student behaviors

Shifts in Attending

Topics other than mathematical thinking dominated the teachers' initial in-the-moment notes (see Table 4). All eight teachers noted that the Zoom environment made it more difficult for the students to collaborate with each other and slowed down the modeling process. Ann's remark (see Table 4) was representative of the observations of the eight teachers since students in both groups seemed to be affected by the Zoom environment in a similar way. Group dynamics and student interactions were also noted by all of the teachers, especially how the students worked with each other (e.g., Alicia's comments in Table 4) and the roles and status of each student in the group (e.g., Alexis's comments in Table 4). In addition, both teachers who enacted the modeling activities were very self-conscious and critical when reflecting on their role as a facilitator (see remarks from Katherine and Brian in Table 4), whereas their colleagues were generous with their evaluations (see remarks from Eric and Ann in Table 4).

Only three of the eight teachers (Ann, Brian, and Sarah, all in Group 2) attended to *mathematical thinking* in their in-the-moment notes, and their focus was limited to the *breakthroughs* during the students' modeling process. One of the three teachers, Sarah noticed that Liam was the first to discuss assigning points to each of the factors and then "totaling them up," and "eventually led the group to a scaling/weighted ranking." Ann and Brian made similar comments about how the students came up with a weighted ranking system (see Table 5).

After I started to use Jacobs et al.'s (2010) framework and discussion prompts to facilitate the teachers' professional noticing experience, their verbal and written responses shifted to focus on students' mathematical strategies (see Table 5). This was especially obvious when I directed the teachers to focus on the "micro-case video clips" that included key dialogues during which students' mathematical thinking was the most visible.

In particular, teachers provided more details of student thinking related to several of the most important aspects of mathematical modeling (see Table 5). Teachers in both groups noticed the blockages their students experienced during the modeling process. For example, in Group 2, Ann was the first to point out that failing to consider all the relevant factors simultaneously while focusing on the one or two factors each student considered as important to themselves was the biggest

Table 4
Sample Responses of Attending to Student Strategies – Non-Mathematical Thinking Topics

Codes	Sample Responses
Zoom Environment	"It would have been a lot more collaboration …. They did a lot of solo work. I believe this would have been different if they had been allowed to perform this task in person." (Ann, group discussion)
Group Dynamics and Student Interactions	"You could tell that the students weren't familiar with each other. They were hesitant to collaborate." (Alicia, in-the-moment notes) "I'm going to pick Liam and I will say 'leadership,' James I'm going to say 'confidence' for sure and Emma, I'm not sure what to say about Emma to be honest, because I mean she was kind of lost and confused the whole time, but she did have some strong points." (Alexis, in-the-moment notes)
Teacher's Role as a Facilitator	"I helped them out with some very big hints, and I feel I handed them the answer." (Brian, in-the-moment notes) "He [Brian] is ensuring that the students were able to explain why and how they come to the conclusions." (Ann, in-the-moment notes) "After that I kicked myself a little because I shouldn't throw that word out there." (Katherine, in-the-moment notes) "She [Katherine] did a good job of asking questions to draw out student thinking in regard to creating a model for the data." (Eric, in-the-moment notes)

Table 5

Sample Responses of Attending to Student Strategies – Mathematical Thinking

Codes	Sample Responses
Breakthroughs	"The biggest kind of revelation is when Liam brought up the weighted system all on his own." (Brian, in-the-moment notes) "They begin using a weighted system, assigning points to each option such as 1 for mostly *open* business, 5 for *mixed*, and 10 for *mostly closed*." (Ann, in-the-moment notes)
Blockages	"They geared all their results towards just those one or two few things … instead of looking across the broad." (Ann, group discussion) "One keyed in on one factor, the other one keyed in on another factor, and then they didn't really start combining the factors until they started actually talking and engaging." (Brian, group discussion)
Model-Reality Mismatches	"They changed a few of the weights for Client 1 and ended up with Idaho and New Mexico being number one and two, but with Brian's help, quickly realized that their number two actually only provided virtual school! Liam stated that they would need to change their weighting scale so that wouldn't happen … (Ann, group discussion) "She [Olivia] looked at the data and said it was steady over a certain interval, then she noticed there was a jump in that rate of change. It increased or became larger than it was during that earlier interval." (Eric, semi-structured interview)
Multiple Solutions/ Between-Model Mismatches	"Liam stated that they would need to change their weighting scale so that wouldn't happen, and James stated maybe they could just toss those states out!" (Ann, group discussion)
Local Conceptual Development	Majority rule → considering factors one at a time and separately → combining/adding/summing scores on each factor → from an unweighted system to a weighted system (Brian, Ann, Sarah, and Alexis, group discussion)

blockage at the beginning stage of the students' modeling process. Brian totally agreed: "Yeah, I think you're right Ann, that's really good" (see remarks from Ann and Brian in Table 5). Teachers in Group 1 also recognized that Katherine's students encountered a blockage when they failed to find a common multiplier that they could plug into the exponential function formula.

Teachers in both groups also noticed that students were able to discover *model-reality mismatches* when engaging in self-evaluation. Two examples are given in Table 5. One was from Ann who noted that Liam and James discovered that their initial weighted ranking system ranked a state (i.e., New Mexico) that offered only virtual schools as the second-best state, which was in contradiction with the client's priority to move to a state that offered in-person education. The other example was from Eric who noticed that Olivia was not able to reconcile what she was observing in the data, "steady over a certain interval" but "then there was a jump in the rate of change," with the linear model in her mind (see Table 5 for Eric's complete remark).

Since there is often more than one path to solve a modeling problem, it was not surprising that teachers in this study noticed *multiple solutions* proposed by different students (see Table 6). For example, teachers in Group 1 noted the differences between Lily's exponential solution and Olivia's linear solution. Eric during his interview with me commented that Lily used the term "percentage increase" between two consecutive numbers and "that's the basis of exponential growth, right? Exactly is a certain percentage increase each unit of time, and that just keeps building at that same growth rate." When it came to Olivia, Eric noted that "one thing that struck me was how she said, going from 55 to 237, which looked at the data from Day 41 to 45. She multiplied by 4 and then added 17" (i.e., a linear function).

Teachers in Group 2 also noticed some different strategies from Brian's students. For example, Ann pointed out that when Liam and James found out that their initial weights did not satisfy a client's request of in-person education, Liam proposed to reassign weights, whereas James proposed to toss out the states that offered only virtual schools (see Ann's remark in Table 5). In addition, teachers in Group 2 noticed the disagreement between James and Liam when they discovered that both South Dekota and Texas received a score of 11. While Liam proposed to have the two states both ranked as number three, James proposed to use "vaccine availability" as a "tie breaker."

Table 6
Student Approaches or Local Conceptual Development Trajectory for the Tasks

Group & Video #	Task	Student Approaches or Local Conceptual Development Trajectory
1	*COVID-19 – Where would we be now??* (see Appendix A)	1. Applying additive reasoning to data or fitting data with a linear function, 2. Looking for and recognizing the new patterns in small chunk of data, 3. Finding a common ratio or multiple that represents the entire dataset by averaging and applying multiplicative reasoning to the data/formally applying the exponential function to the dataset.
2	*We're Moving!* (see Appendix B)	1. Majority rule, 2. Consider factors one at a time and separately, 3. Combine/add/sum scores on each factor, 4. From an unweighted system to a weighted system, based on prioritization of factors.

Teachers in Group 2 were able to collectively map out a *local conceptual development* trajectory of student thinking in a similar fashion that a Jigsaw puzzle would have been built (see Table 6). Each teacher contributed at least one piece of the puzzle. In particular, teachers characterized this particular group of students' modeling process into four stages, and delineated how, from stage to stage, students gradually developed mathematical concepts such as ranking, aggregating ranked quantities, and weighting ranks:

1. *Majority rule*: Sarah and Ann noted that initially students were comparing their own rankings with the others. Liam found that James's and Emma's rankings were more like each other's than to his: "I will make mine look more like you guys' because you guys' are kind of similar."

2. *Consider factors one at a time or separately*: All four teachers noted that at this stage, students encountered a blockage where they did not know how to consider multiple factors simultaneously and what mathematical tools or concepts they could use.

3. *Combine/add/sum scores on each factor*: Ann, Brian, and Sarah noted that Liam was the first one to consider a point system by stating:

   > If we are going to do the point thing like ranking it 1 to 3 or 1 to 8, you want the lower numbers, except for on vaccine availability, yeah, you want lower numbers so like a perfect score would be a 3.

   Then he calculated New Mexico's score as "1 and 6 and 1, that'd be eight."

4. From an unweighted system to a weighted system: All four teachers recognized this stage, especially how Liam figured out the idea of weighting on his own and proposed to double or triple the scale for the business factor.

Noticing students' local conceptual development helped teachers develop more confidence in their students' ability to tackle challenging mathematical concepts, for instance, in Alicia's word, "I was in awe." Alicia reflected during the exit interview: "I should expect more from my students. I learned that. I think I have handicapped them because by assuming that they wouldn't be able to do something without even trying it." Sarah also ended her interview with me on a very positive note: "It gave me more confidence as the teacher being able to introduce modeling, and then it gave me more confidence, I guess, in the students."

Shifts in Interpreting

All eight teachers' in-the-moment notes contained limited interpretations of students' thinking. When the teachers attempted to respond to the prompts under the *interpreting* section, they tended to provide *snap evaluations* with no or very vague evidence of student thinking. An example from Eric is shown in Table 7. From this remark, he was taking a more evaluative stance than an interpretive stance.

All eight teachers showed more confidence in interpreting student thinking after they had an opportunity to interact with each other. When they tried to make sense of the students' evolvement during the modeling process, teachers in Group 2 made deeper inferences beyond describing the four stages mentioned above. In particular, they hypothesized that disagreement among students (i.e., between-model mismatches) created the first necessity for them to move from Stage 1 to Stage 2 and to Stage 3. The second necessity occurred when there was a failed validation from model to the goal or criteria (e.g., model-reality mismatch). This mismatch created the second necessity that propelled the students to move from Stage 3 to Stage 4. Brian summarized the mechanisms that facilitated the students' modeling process as "necessity is the mother of invention" (see Table 7), in other words, mismatches had driven the local conceptual development observed in the students. Group 1's teachers also mapped out a local conceptual development trajectory for their students but did not make conjectures about what was the driving force behind students' conceptual development.

Teachers in Group 1 instead focused on looking for the root cause of the students' initial struggles and attributed the blockage the students encountered to inflexible thinking due to rule-based pedagogy. For example, Katherine diagnosed her students' way of thinking and concluded that their struggle during the modeling process was due to how the exponential function was taught to them (see Katherine's remark in Table 7).

Table 7
Sample Responses of Interpreting Student Thinking

Codes	Sample Responses
Snap Evaluation	"Lily does a good job of explaining her thinking and explaining her process and coming up with a model in describing how she came up with the growth rate." (Eric, in-the-moment notes)
Mismatches driving local conceptual development	"Necessity is the mother of invention." (Brian, group discussion and semi-structured interview)
Inflexible thinking due to rule-based pedagogy	"They were taught theoretically [rule-based] more than with models. While they could identify COVID as an exponential growth function, they had not covered scatterplots and therefore, were struggling with a curve of best fit that did not exactly fit the data." (Katherine, final written response) "They do not know what to do when presented with an imperfect data set that doesn't fit an equation perfectly ... there is not a lot of statistics or line of best fit, regression line type of thought [in the curriculum]" (Eric, semi-structured interview)

Carrie, in her final written response also stated, "They were too focused on making sure that it fit exactly right." During group discussion, both Carrie and Alicia commented on the "it's got to work" type of mentality. It was used by the teachers to explain student behaviors. For example, none of the three students thought about graphing the data to observe its overall shape. Instead, all three students seemed to be at a loss at first, followed by checking their individual calculations. They hoped to find some errors that could help them reconcile the imperfect fit between model and data. Similarly, Eric, in his individual interview with me, also interpreted students' way of thinking as an indication that the school mathematics curriculum lacks a modeling or statistical approach (see Table 7).

Shifts in Deciding How to Respond

None of the teachers provided any suggestions for next problems or next steps in their in-the-moment notes. They reported either failing to find any basis to respond or not being able to understand the task. By the end of the study, all eight teachers made recommendations for the next steps. Their responses came from two distinct perspectives. The first focused on strategies that connected the mathematics content with pedagogy and tended to be rooted in student thinking, i.e., "thinking-based strategies." The second perspective tended to be rooted in student behaviors or roles, not in student thinking, i.e., "behavior-based strategies."

Ann's next steps for Liam and Brian's next steps for Emma were typical thinking-based strategies. Ann suggested having Liam solve another task in the modeling sequence on ranking to see if he was able to generalize his thinking to "other scenarios … not move into a city [during the COVID pandemic]." Ann justified her decision based on Liam's comfort level with his group's modeling task, especially how Liam was able to adjust his original weighting scheme for Client 1 to fit Client 2: "Even with Client 2, he did not give enough weight to his first factor, but he quickly realized it once he chose the best states and then stated he should have weighted heavier." Brian suggested using the same task or a similar one for Emma. He justified his decision by diagnosing Emma's blockage during the modeling process: "She understood ranking something the sixth highest, etc., … but let's apply a point value to it, then tally up our point value." Brian further explained that in his opinion, Emma was not as far behind the two boys as it seemed, and she was "on the verge" of coming up with a number system. But she fell short on "assigning a numerical value to each of the states that would lead her to a total score."

Most strategies proposed by Group 1's teachers were thinking-based. For the students as a group, Alicia, Katherine, and Eric all proposed providing students with a graph as a visual aid to help students see the overall shape or trend of the data. Katherine and Eric went more in-depth to propose an entirely alternative approach to the task. Katherine stated in her written response: "Instead of giving them the numbers, if we gave them a graph they would see the scatterplot, they could probably develop a curve of best fit out of it and that would have been more appropriate for them." Katherine's decision was based on her noticing of students' struggle with choosing which function, linear or exponential, should be used to represent the dataset, when they could not find a common multiplier.

Teachers also proposed behavior-based strategies such as continuing to use cooperative learning because the students worked well together and showed chemistry with each other (Sarah) and putting Josh in one-on-one tutoring situation where his thinking could be more visible because Josh was reserved during the group work (Katherine, Eric).

Discussion

Jacobs et al.'s (2010) framework of professional noticing as a scaffolding tool, combined with the video club format, played an important role in shifting the teachers' attention to student thinking. Initially, five of the eight teachers completely focused on other aspects of the videos even if they were asked to focus on student thinking. For example, teachers were quick to notice the limitations of the Zoom environment such as no shared, physically tangible space for students to observe each other's work and collaborate. Both Brian and Katherine were critical and self-conscious of their own performance as a facilitator. Group dynamics and student interactions were also discussed, but the attention was given mostly to assigning leaders and followers. This finding is consistent with that of Van Es and Sherin (2008), who also found that teachers' initial thoughts when discussing a video tended to be spread over multiple aspects of the learning environment, and deliberate training was needed to guide teachers to specifically attend to students' mathematical thinking and detailed strategies.

The results of this study also support that professional noticing in the context of modeling has some special aspects such as noticing students' local conceptual development as related to the modeling task they were solving and the wide range in student thinking (Lesh & Doerr, 2003). Teachers were able to use their knowledge about the modeling process to identify blockages and mismatches that were crucial to push students to a higher stage, a phenomenon also observed in Lesh and Harel (2003). While acknowledging that meaningful and lasting development of overall professional noticing skills often requires years of participation in professional development (Jacobs et al., 2010), I found that all eight teachers demonstrated growth in noticing skills related to the one video they observed within a short amount of time under the guidance provided by Jacobs et al.'s framework and discussion prompts. In addition, two types of responses to student thinking were also proposed, thinking-based strategies and behavior-based strategies. This finding was also consistent with that of Jacobs et al., who pointed out that there can be many effective ways for teachers to build on students' existing understandings, including both thinking-based strategies such as proposing a new problem, asking students to compare strategies, probing students' underlying reasoning, and behavior-based strategies, including pairing students up so that they can help each other.

Earlier I noted that prior to designing their own tasks and engaging in professional noticing activities, I engaged teachers in solving several MEAs in small groups (i.e., Group 1 and Group 2), then analyzing the MEAs and reflecting on their group's modeling process. These activities facilitated their development of theoretical knowledge of modeling and may help explain the growth observed in the teachers within a short amount of time. In fact, teachers' development of knowledge about mathematical modeling happened along with their growth in noticing skills. Due to limited space, a report on teachers' growth in theoretical knowledge about mathematical modeling will be presented elsewhere. However, the results of this study should also be treated with caution because all of the teachers only watched and analyzed student thinking in one video (i.e., either Video 1 or Video 2). Given their limited knowledge about mathematical modeling before this study, the professional development sessions implemented need to be iterated with multiple modeling tasks, including tasks beyond MEAs, for developing strong overall professional noticing skills.

Like all research endeavors, this study has some limitations. First, because of the restrictions due to the COVID pandemic, each group of teachers enacted their modeling task with a small group of three students via Zoom. This compromise made it impossible for me to observe teachers' selective attention in a busy and more complex whole-classroom environment (Sherin et al., 2011). On the other hand, teachers in this study were not familiar with mathematical modeling, and they had never been asked to engage in professional noticing of student thinking in the context of mathematical modeling prior to the study. Therefore, teachers learning to engage in professional noticing could benefit from first focusing on a small group of students before they are asked to observe a whole class of students. However, the benefits for teachers to observe a small group of students as an intermediate step before observing a whole class of students have not been examined in the literature of professional noticing. Future researchers may design experiments to provide empirical evidence of the benefits of adding this intermediate step. Second, some key developments in student thinking during the modeling process seemed perplexing to the teachers, for example, how Liam was able to come up with the idea of weighting by multiplying on his own and how Olivia, who had no prior exposure to exponential functions, caught up with her group. Future researchers are recommended to interview students as well as their teachers to triangulate interpretations of student thinking from both perspectives, especially when a student seems to show local conceptual development.

Despite the short duration of the professional noticing training during which each group of teachers only worked with one video, the study contributes to our understanding of some of the special characteristics of noticing in the context of modeling including noticing blockages, mismatches, multiple solutions, and local conceptual development. Future researchers may consider using the information in this study to design formal diagnostic or assessment tools for professional noticing in the context of modeling. The study also provides preliminary evidence that Jacobs et al.'s (2010) framework together with discussion prompts are effective scaffolding tools for generating productive conversations among teachers about student thinking and for developing teachers' professional noticing skills. Future researchers may adapt the training sessions in this study and implement multiple cycles of these sessions with a sequence of MEAs or other types of modeling tasks. In the past, modeling as a means to developing professional noticing has not been fully taken advantage of. This study shows that the modeling process can generate rich student thinking and therefore is a fertile ground for developing professional noticing. Finally, in spite of the various limitations of the Zoom environment pointed out by the teachers in this study, this online synchronous environment also seems to be a viable alternative tool for research, especially during a time when in-person observation is not possible. Future researchers are recommended to continue to explore the pros and cons of the Zoom environment as a medium for research in mathematics education.

References

Alwast A., & Vorhölter, K. (2022). Measuring pre-service teachers' noticing competencies within a mathematical modeling context–an analysis of an instrument. *Educational Studies in Mathematics*, *109*, 263–285. https://doi.org/10.1007/s10649-021-10102-8

Anhalt, C., & Cortez, R. (2016). Developing understanding of mathematical modeling in secondary teacher preparation. *Journal of Mathematics Teacher Education, 19*(6), 523–545. https://doi.org/10.1007/s10857-015-9309-8

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modeling process. *ZDM: The International Journal on Mathematics Education, 38*(2), 86-95. https://doi.org/10.1007/BF02655883

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to other subjects: State trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*, 37-68. https://doi.org/10.1007/BF00302716

Cetinkaya, B., Kertil, M., Erbas, A. K., Korkmaz, H., Alacaci, C., & Cakiroglu, E. (2016). Pre-service teachers' developing concepts about the nature and pedagogy of mathematical modeling in the context of a mathematical modeling course. *Mathematical Thinking and Learning, 18*(4), 287-314. https://doi.org/10.1080/10986065.2016.1219932

Charmaz, K. (2003). Grounded theory: Objectivist and constructivist methods. In N. K., Denzin & Y. S. Lincoln (Eds.), *Strategies for qualitative inquiry* (2nd ed., pp. 249-291). Sage.

Doerr, H. M. (2007). What knowledge do teachers need for teaching mathematics through applications and modeling? In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modeling and applications in* *mathematics education: The 14th ICMI Study, New ICMI Studies Series no. 10* (pp. 69-78). Springer.

Doerr, H. M., & English, L. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education, 34*, 110-136. https://doi.org/10.2307/30034902

Fereday, J., & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods, 5*(1), 80-92. https://doi.org/10.1177/160940690600500107

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modeling process. *ZDM, 38*(2), 143-162. https://doi.org/10.1007/BF02655886

Goodwin, C. (1994). Professional vision. *American Anthropologist*, *96*(3), 606-633. https://doi.org/10.1525/aa.1994.96.3.02a00100

Han, C., Kim, H., Kwon, O. N., & Lim, W. (2023). Exploring changes of mathematics teachers' noticing in a video club: Identifying turning points. *International Journal of Science and Mathematics Education, 21*, 835-861. https://doi.org/10.1007/s10763-022-10251-z

Jacobs, V. R., Lamb, L. L. C., & Phillip, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202. https://doi.org/10.5951/jresematheduc.41.2.0169

Lesh, R. A., & Doerr, H. M. (2003). *Beyond constructivism: Models and modeling perspectives on mathematical problem solving, learning, and teaching*. Lawrence Erlbaum Associates.

Lesh, R. A., & Harel, G. (2003). Problem solving, modeling, and local concetpual development. *Mathematical Thinking and Learning*, *5*(2-3), 157-189. https://doi.org/10.1080/10986065.2003.9679998

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge

Patton, M. O. (2002). *Qualitative research and evaluation methods* (3rd ed.). Sage.

Piaget, J. (1950). *The psychology of intelligence* (Translated by M. Piercy & D. E. Berlyne). Routledge.

Sherin, M. G., Jacobs, V. R., & Philipp, R. (2011). Situating the study of teacher noticing. In M.G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 3–12). Routledge.

Sherin, M. G., & Van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*(1), 20-37. https://doi.org/10.1177/0022487108328155

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert, & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107-150). Cambridge University Press.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques* (2nd ed.). Sage.

Tripp, D. (2011). *Critical incidents in teaching: Developing professional judgement*. Routledge.

Ulusoy, F., & Çakıroğlu, E. (2020). Exploring prospective teachers' noticing of students' understanding through micro-case videos. *Journal of Mathematics Teacher Education, 24*, 253-282. https://doi.org/10.1007/s10857-020-09457-1

Van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24*(2), 244–276. https://doi.org/10.1016/j.tate.2006.11.005

Wess, R., Klock, H., Siller, H., & Greefrath, G. (2021). *Measuring professional competence for the teaching of mathematical modeling: A test instrument*. Springer.

Appendix A

Group 1's Task

Part I: Where would we be now???

As a scientist, you are concerned with how quickly the cases count, and deaths increased in the first three months of the pandemic. Fortunately, after the first spike we saw a drop in the rate because of restrictions put on by the government and people's caution in general.

1. Use the data below (shown in 4-day jumps for the first 109 days) to find a mathematical model that fits the number of cases over the first 109 days. Then use the second column to create a model that will fit the data for the number of deaths in the first 71 days (After the first death).

2. It has now been 444 days since the first case was reported in the United States. Using your models, predict how many cases we could have expected to see if it continued at the same rate as the first days.

3. What do you notice about the difference between the expected number of cases and the actual number? What do you think led to the difference?

U.S. CASES	
Day#	Total Cases
1	1
5	5
9	6
13	11
17	12
21	13
25	14
29	14
33	16
37	17
41	55
45	237
49	782
53	2978
57	9169
61	34898
65	86693
69	165987
73	289066
77	413507
81	544185
85	652591
89	769684
93	887858
97	1000785
101	1115946
105	1216209
109	1320155

Note. Part II of the task and a background reading piece are omitted. I am happy to share the complete task upon request.

Appendix B

Group 2's Task

You have just started your new job at a relocation consultant company. Your company specializes in making recommendations to people who are looking to move, but are not sure where, and so they hire you. This is a new business, and your boss has tasked you to develop an advising system for choosing places for the clients to live.

Because of the recent COVID-19 pandemic, there have been several clients who want to relocate to another state. The clients are primarily interested in Covid related concerns: virtual & in- person schools, total Covid cases, vaccine availability, and business availability. We have gathered data to and advise from 8 varying states throughout the US.

We have two clients that are looking for your recommendation for where to move. Below are their letters of preferences and concerns.
1. Develop a rating system for comparing COVID-19 in the varying states. Be sure your system will really help the agency evaluate places, even those not listed below.
2. Write two letters for the consultant company with your recommendations for each of our new clients. You should put the states into 3 groups: the best choice states, the second-best states, and the worst states.
3. You should explain to the company how your rating system works and why it is a good one.

COVID-19 DATA

State	Virtual/In-Person School	Total # of COVID-19 Cases (per 100,000)	Vacine % Availability	Mask Mandate	Businesses
Alabama	Either Virtual/In-Person	10,438	11.0%	Mandatory	Mostly Open
California	Virtual Only	8,978	12.1%	Mandatory	Mostly Closed
Georgia	Either Virtual/In-Person	9,675	10.5%	No Restrictions	Mostly Open
Idaho	In-Person Only	9,947	13.2%	No Restrictions	Mostly Open
New Mexico	Virtual Only	9,056	19.3%	Mandatory	Mixed
New York	In-Person Only	9,672	12.9%	Mandatory	Mostly Closed
South Dakota	Virtual Only	13,109	17.9%	No Restrictions	Mostly Open
Texas	In-Person Only	9,476	10.7%	No Restrictions	Mostly Open

Client #1
Dear Relocation Consultant Specialist,

My Husband and I are retiring in several months and will be moving. It will just be the two of us, as we have no children. We are concerned about COVID in the various area, and how safe the areas are in dealing with the pandemic. Neither of us have had the vaccine, but would like too. We really want to enjoy our retirement, and safely partake activities in the community. Thank you so much!
Sincerely,
Mr. & Mrs. James

Client #2
Dear Relocation Consultant Specialists,
My family are looking to move very soon. We have been out of work because of COVID, and need to move somewhere else. My wife and I have 3 children who will be accompanying us. The kids have really struggled in the online learning environment, and we want to be able to go out and have a family dinner together. Of course, we would like to like to be as safe as possible concerning Corona, but we want to live our lives with less regulations.

Sincerely,
Mr. Sanders

Note. A background reading piece is omitted. I am happy to share the complete task upon request.