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A Framework for Time and Covariational Reasoning

Kevin C. Moore

Covariational reasoning has emerged as a productive construct to characterize students' mathematical development. Researchers have illustrated its importance for major middle, secondary, and undergraduate mathematical concepts, including rate of change, accumulation, and modeling. Within this line of work, several researchers have indicated differences between experiential and conceptual time with respect to the covariational relationships students construct. I draw on this body of literature and return to Piaget's perspective of time to provide a framework for the role of time in students' covariational reasoning. The framework also clarifies the nature of the multiplicative objects underlying students' covariational relationships. To illustrate the framework and capture its emergence from second-order models of students' mathematics, I describe the framework as it relates to students' engagement in a task.

> "We are far too readily tempted to speak of intuitive ideas of time, as if time, or for that matter space, could be perceived and conceived apart from the entities or the events that fill it" (Piaget, 1970, p. 1).

Time has been a topic of contemplation for researchers and philosophers for centuries, and ontological and epistemological considerations of time are certainly not restricted to the academy. The mere question of "What is time?" can provoke a lively conversation among most groups of individuals, and numerous pop culture pieces have built their plotlines around the fabric of time and its manipulations. The prevalence of time as a subject of rumination is unsurprising, given its inseparable role from existence and experience. Kant (1781/2003) considered time and its properties to be so ubiquitous as to be given a priori.

Kevin C. Moore *is a professor of mathematics education at the University of Georgia. His primary research focuses on student and teacher meanings, particularly as they*

Countering Kant's perspective, Piaget considered time to be a constructed concept subject to the mental operations defining a person's experience (Piaget, 1954; von Glasersfeld, 1995, 1997). He dedicated several studies to developing conceptual models of that construction (e.g., Piaget, 1954, 1970), ultimately determining that the mental operations involved in coordinating at least two objects' motions are fundamental to an individual's construction of time. Piaget concluded that concepts of time are inseparable from the mental operations defining space, motion, and objects; not only is time a constructed concept, but it is also the product of coordinating mental operations involved in the construction of other concepts (Piaget, 1970; von Glasersfeld, 1984). Ludwig and Luciano expressed as much, "We cannot compare a process with the 'passage of time' – there is no such thing – but only with another process" (2021, p. 240).

Building on the work of researchers who have alluded to covariational reasoning—which involves the variation and motion of objects—being connected to conceptions of time, I return to Piaget's (1970) conceptual models for time to further develop the role of time in students' covariational reasoning. In doing so, I elaborate on the constructs of *experiential time* and *conceptual time* (Castillo-Garsow, 2012; Thompson & Carlson, 2017) to provide a framework for characterizing students' covariational reasoning in relation to concepts of time. Reflecting its empirical roots, I illustrate the framework by describing a task designed to provide insights into the role of time with respect to students' covariational reasoning. The task and its description are informed by second-order models of students' mathematics developed during a series of teaching experiments (Steffe & Thompson, 2000), and I mention salient aspects of those models to connect the task and framework.

Covariational Reasoning and Time

The connection between time and the motion or variation of objects has been indicated within work on students' covariational reasoning (e.g., Johnson, 2015b; Paoletti & Moore, 2017; Patterson & McGraw, 2018; Stalvey & Vidakovic, 2015; Thompson & Carlson, 2017). Covariational reasoning—the cognitive activities involved in reasoning about how quantities vary in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998)—is an emergent area of research within the landscape of quantitative reasoning. Researchers exploring covariational reasoning have illustrated its importance for the learning of middle, secondary, and undergraduate mathematics concepts (Byerley & Thompson, 2017; Carlson et al., 2002; Ellis, 2011; Ellis et al., 2015; Johnson, 2015a, 2015b; Moore, 2014; Paoletti et al., 2023; Thompson et al., 2017), with other researchers identifying its importance beyond mathematics classrooms and contexts (Gantt et al., 2023; Rodriguez et al., 2019; Sokolowski, 2020; Yoon et al., 2021). For example, Yoon et al. (2021) illustrated its importance for citizens' assessment of COVID-19 pandemic data representations.

With respect to relationships between time and covariation, researchers have primarily focused on time as a parameter (Keene, 2007; Kerrigan, 2023; Kertil et al., 2019; Paoletti & Moore, 2017; Patterson & McGraw, 2018; Stalvey & Vidakovic, 2015). A focus on time as a parameter is sensible given the ubiquitous role of time in experience and mathematical parameterization. Accordingly, these researchers have focused on the extent to which time is held implicitly or explicitly in students' minds as they construct and reason about relationships between quantities. For instance, Patterson and McGraw (2018) explored students' meanings in the context of dynamic situations and their graphing of quantitative relationships that did not include elapsed time—quantified time as measured or displayed on a clock—as a graphed quantity. Relatedly, Paoletti and Moore (2017) explored how graphing experiences with quantitative relationships not explicitly involving elapsed time can create an intellectual need for time as a parameter. Taking a different approach, Stalvey and Vidakovic (2015) focused explicitly on students constructing relationships between elapsed time and two other quantities and their subsequent construction of a relationship between those two quantities. In each of these studies, the authors' framing of time focused on whether or not a student or task identified time as an explicit quantity.

Some researchers have drawn on notions of conceptual and experiential time, which Castillo-Garsow (2010, 2012) and Thompson (2011, 2012) introduced to characterize students' covariation. Having roots in Newtonian mathematics and loose connections to Piaget's (1970) framing of time, Thompson (2012) described experiential time as "felt time that [passes]" in an experience, while conceptual time is "Not time on a clock, but an imagined, smoothly changing, quantified time—a measured duration that grows in extent" (p. 147). Modeling students' meanings for growth, Castillo-Garsow (2010) added that experiential time is that which passes for oneself and is thus always in progress and "inherently continuous" (p. 202). By inherently continuous, Castillo-Garsow was not referring to conceptions of measured time, nor was he speaking to recalled, subjective perceptions of distinct events and judgments of temporal experience. Rather, he was referring to our inability to escape the present, and that experiential time is that which is always passing. We are continuously constrained by experiential time. Experiential time is a lived and figurative form of time, whereas conceptual time is an operative form of time. Thompson and Carlson (2017) added that conceptual time as an image of measured duration does not necessarily imply the individual has in mind actual timed values, but rather that the individual imagines "a quantity as having different values at different moments, and envision[s] that those moments happen continuously and rhythmically" (p. 437). This framing of conceptual time underscores the centrality of "images" or "imagery" in conceptions of time, which refer to the re-presentation of experience and are not as well-defined as operations. Images or imagery incorporate a plethora of experiential aspects, including kinesthesia, affect, and sensory sensations (Thompson, 1994; Thompson et al., 2024).

Thompson and Carlson's descriptions of conceptual and experiential time also underscore that they are akin, but not identical to the aforementioned parametric distinctions. Whereas parametric distinctions frame time as an implicit or explicit attribute in and of itself, the experiential and conceptual distinctions position time as an emergent, intrinsic property of covariation that differs based on the covariation conception. Said another way, parametric distinctions are with respect to the extent a task or reasoner explicitly identifies time as a relevant quantity. A task either states time as an explicit quantity or not, or an individual explicitly identifies time as a relevant quantity or not. The distinction between conceptual and experiential time concerns an individual's conception of a relationship, regardless of the extent to which time is explicitly identified as a relevant quantity. Conceptual and experiential time are organic to quantities' covariation, and thus the distinction between the two is situated in how a phenomenon's attributes are conceived, which mirrors Piaget's (1970) positioning of time.

Piaget, Time, and Co-Variation

Piaget considered space and time to be inseparable, with time being an emergent property of the co-ordination of simultaneous positions and the co-ordination of successive, spatial states (i.e., co-seriation). Whereas Fraisse (1984) referred to time concepts as succession and duration, Piaget referred to these co-ordinations as simultaneity and succession (with displacement), respectively, and he positioned their development and, hence, the development of time as occurring in the context of motions with different velocities. Given two objects in motion, time concepts emerge from both the co-ordination of one object's position with the other object's simultaneous position in combination with the co-ordination of their successive respective positions in space.

Piaget's view of time as rooted in conceptions of space and motion reflects his stance that concepts arise from the coordination and abstraction of mental actions. To Piaget, our temporal experience and memory of a situation are constructions subject to mental actions. We transition from intuitive conceptions of time to operative conceptions as we develop operative ways for organizing experience in place of experiential or figurative organizations. An individual lays the basis for operational or conceptual time through the construction and abstraction of the operations involved in the co-ordinations of simultaneity and succession (Piaget, 1970). For example, given two objects in motion, an individual lays the foundation for operational or conceptual time as they transition from conceiving motion in terms of the relative perceptual location of each object (e.g., the ending place of each object's movement) to coordinating the simultaneous displacement of objects (e.g., the entire span covered by each object's movement).

Piaget (1970) formalized the construction of simultaneity and succession of multiple events as shown in Figure 1. Figure 1 is not an exact reproduction of Piaget's model of events (see Piaget, 1970, p. 264). I change some symbolic conventions of his model to better connect to the models I provide below and be more consistent with current symbolic conventions. Here, X_0 represents the initial state of event X. I use the term *state* to capture the generic conception of some event having a condition at an experiential moment that can be indexed with respect to the event's condition at other experiential moments. These different experiential moments necessarily have temporal components, but a conception of an event's different conditional states need not foreground those components beyond understanding that they occur at different moments of experience. For example, a golfer could recall, index, and order each event in which they shot a personal best score without an explicit conception of the time elapsed between those events. In terms of *event,* event X could be any attribute of an object or phenomenon. With respect to Figure 1, A, B, and C could represent a person's height, weight, and age, respectively. $X_{\#}$, $X_{\#+1}$, $X_{\#+2}$, and so on represent successive conditional states of event X with $\Delta t_{\#}$ representing the duration between two successive states (e.g., #-1 and #). Thus, A_0, B_0, and C_0 represent the person's height, weight, and age, respectively, at some chosen initial state. A_1, B_1, and C_1 represent the person's height, weight, and age, respectively, at some subsequent state, and Δt_1 represents the elapsed time between those states. Piaget used $\downarrow^{o}$ to link states of events occurring simultaneously (e.g., an object's weight and height). One explanation for Piaget's use of $\downarrow^{o}$ is that it can be thought of as a null vector. As opposed to a symbol that might imply a transformation between the state of one event to the corresponding state of another event (e.g., A_1 and B_1), a null vector has zero magnitude and no particular direction and thus $\downarrow^{o}$ emphasizes the states as occurring or existing independently, yet simultaneously. This is consistent with Piaget's (1970) model that captures the multiplicative basis of co-seriation, in which events are united to form a *multiplicative object*—the cognitive uniting of multiple attributes so that an object is simultaneously all of them (Inhelder & Piaget, 1964). Saldanha and Thompson (1998) identified that constructing such an object is fundamental to the covariation of quantities, a point I return to below.

Figure 1. *A model of simultaneity and succession as modified from Piaget (1970, p. 264).*

$$
\begin{array}{ccccccccc}
\mathbf{A_0} & \xrightarrow{\Delta t_1} & \mathbf{A_1} & \xrightarrow{\Delta t_2} & \mathbf{A_2} & \xrightarrow{\Delta t_3} & \mathbf{A_3} & \xrightarrow{\Delta t_4} & \cdots \\
\downarrow\mathrm{o} & & \downarrow\mathrm{o} & & \downarrow\mathrm{o} & & \downarrow\mathrm{o} & & \\
\mathbf{B_0} & \xrightarrow{\Delta t_1} & \mathbf{B_1} & \xrightarrow{\Delta t_2} & \mathbf{B_2} & \xrightarrow{\Delta t_3} & \mathbf{B_3} & \xrightarrow{\Delta t_4} & \cdots \\
\downarrow\mathrm{o} & & \downarrow\mathrm{o} & & \downarrow\mathrm{o} & & \downarrow\mathrm{o} & & \\
\mathbf{C_0} & \xrightarrow{\Delta t_1} & \mathbf{C_1} & \xrightarrow{\Delta t_2} & \mathbf{C_2} & \xrightarrow{\Delta t_3} & \mathbf{C_3} & \xrightarrow{\Delta t_4} & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Drawing on Piaget's model of time and the simultaneity and succession of events, I present three conceptual models of time as it relates to an individual's conception of a phenomenon that entails quantities' magnitudes (e.g., $||x||$, $||y||$, $||z||$, …) varying. To support the presentation of these models, I provide

Table 1 to define the symbols used in each. After briefly defining each model, I elaborate upon the structure and covariational relationship conveyed by each. Before presenting each model, I note two general modifications to Piaget's model (Figure 1). Firstly, Piaget's model does not make explicit the operative aspect of conceptual time, which entails being able to imagine time elapsing forward or backward. In order to differentiate between conceptual and experiential time, the latter of which only flows forward, I use ↔ and →, respectively.

Secondly, I modify Piaget's use of $\downarrow$o to capture various levels of conceptual linkages between quantities' magnitudes. I also modify it to not imply a transformation or cognitive switch of focus between quantities' magnitude. I elaborate on this below.

***Table 1**. Glossary of symbols.*

SYMBOL	DESCRIPTION
$\|\|x\|\|$	The magnitude of quantity x
$\|\|x\|\|_0$	The magnitude of quantity x at its initial state (i.e., state 0)
t_e	Experiential time
$\|\|\boldsymbol{x}\|\|_{t_e}$	The magnitude of quantity x at some moment of flow in experiential time
$t_\#$	The duration of (conceptual) time that elapses between state 0 and state #
$\Delta t_\#$	The duration of (conceptual) time that elapses between state # - 1 and state # (i.e., $t_2 = t_1 + \Delta t_2$)
$\|\|\boldsymbol{x}\|\|_{t_\#}$	The magnitude of quantity x at (conceptual) time $t_\#$
$\|\|x\|\|_\#$	The magnitude of quantity x at state #
$\longrightarrow$ $\longleftrightarrow$	The flow of experiential time or bi-directional nature of conceptual time
⁞ , I , and **I** o	Conceptual linkages between quantities' magnitudes that indicate coupled variation, a connection mitigated by conceptual time, and a multiplicative object, respectively

The first model (Figure 2) conveys a conception tied to experiential time. The second and third models (Figure 3a and Figure 3b) each convey a conception tied to conceptual time. Figure 3a foregrounds the phenomenon as conceived with respect to elapsed time, while Figure 3b involves disembedding the quantities from the phenomenon and elapsed time so that their invariant relationship—the deterministic pairing that exists through linking each quantity's variation—is taken as the object of thought. Here, my use of disembedding is informed by the disembedding operation introduced in number and fractional reasoning research (see Steffe, 2001; Steffe & Olive, 2010). The disembedding operation refers to the mental act of removing a part from a whole while keeping the whole mentally intact, which is critical to the act of constructing and iterating a unit (Steffe, 2001). Regarding Figure 3b, I speak to a more general form of disembedding, in which the quantities are removed or pulled from a phenomenon and its experience while not only keeping the phenomenon intact, but also keeping the covariational properties between them mentally intact independent of the phenomenon and its experience. Such an act is critical to constructing graphs as re-presentations of covariational relationships (Lee et al., 2020; Moore, 2021; Moore et al., 2013).

Figure 2. *Conceiving covarying quantities of a phenomenon with respect to experiential time.*

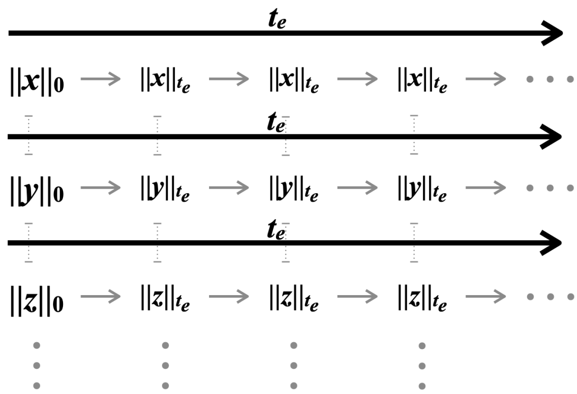
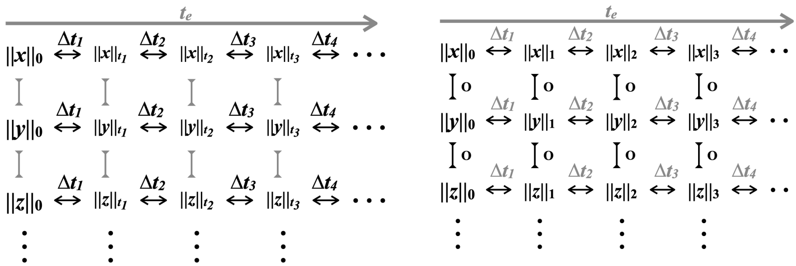


Figure 3. *Conceiving covarying quantities of a phenomenon with respect to (a) conceptual, elapsed time and (b) so that the quantities are disembedded and their invariant relationship is taken as the object of thought.*



(a) (b)

Adopting expression notation and restricting the focus to the two quantities x and y, we can represent Figure 2, Figure 3a, and Figure 3b with $||x||_{t_e} \vee ||y||_{t_e}$, $(||x||_t \vee ||y||_t)$, and $(||x||_\# \wedge ||y||_\#)$, respectively. I use $||x||_{t_e} \vee ||y||_{t_e}$ with $\vee$ (OR) and no parentheses to indicate that when a phenomenon and its constituent quantities are conceived with respect to experiential time, the quantities are both understood as present and varying in experience. They are observed to co-occur, but they are not cognitively linked beyond that. A conception of their relationship involves sequentially recalling the intuitive, in-the-moment experience or flow of each quantity's variation; as I illustrate using an example below, each quantity's variation is constrained to being conceived in experience, and thus, this form of covariation is technically coupled variations. This is captured by the faded and backgrounded link between $||x||$ and $||y||$ in Figure 2, as well as the foregrounding of each quantity's variation with the continuous flow of experiential time, t_e, rather than with durations between each state. Recall that experiential time is not a form of measured time, and thus t_e is used to convey that each quantity and its variation are always conceived in the flow of experience.

I use $(||x||_t \vee ||y||_t)$ and $(||x||_\# \wedge ||y||_\#)$ to indicate a phenomenon and its constituent quantities conceived with respect to conceptual time, whether that be elapsed time or their relationship being disembedded and understood with respect to variations between different indexed states. With respect to $(||x||_t \vee ||y||_t)$, I use parentheses to indicate that the quantities are understood as occurring simultaneously, but I use $\vee$ to indicate that elapsed time is the driver of the relationship such that the two quantities are related through their sharing a relationship with elapsed time. They are understood to covary, but their pairing does not form a multiplicative object with each other. This is captured by the link between $||x||$ and $||y||$ in Figure 3a, which is stronger and more foregrounded than that in Figure 2. But, the link is mitigated by the connection to elapsed time, and this is indicated by a faded connector. With elapsed time as the driver of the relationship, experiential time moves to the background, and durations are foregrounded in its place.

With respect to $(||x||_\# \wedge ||y||_\#)$, I use $\wedge$ (AND) and parentheses to indicate that the quantities are understood as occurring simultaneously and persistently. The quantities' magnitudes are the driver of the relationship, and thus properties of the relationship are understood as strictly defining the two quantities' magnitudes. These properties are understood as being sustained irrespective of figurative aspects of experience or how one steps through durations of elapsed time. In tandem with fading specific durations to the background, this is captured by the link between $||x||$ and $||y||$ in Figure 3b. This solid link indicates their simultaneous and persistent co-existence so that their covariation is defined precisely by their simultaneous variations. I also note that the shape linking $||x||$ and $||y||$ in Figure 3a and Figure 3b is different than that in Figure 2 to indicate an operative association between the two magnitudes that necessarily entails a conception rooted in conceptual time (Piaget, 1970). Despite the bi-directional nature of this operative association (Paoletti, 2020; Paoletti & Moore, 2017), I choose not to use arrows for such a link so as not to imply a transformational or motion image. Relatedly, I use O with this link (e.g., ) to reflect Piaget's perspective that states of events paired to form a multiplicative object (e.g., an object's weight and height) can be thought of as connected by a null vector due to their simultaneity.

Returning to the relationship between constructing a multiplicative object and covariational reasoning, Saldanha and Thompson (1998) noted that merely thinking of two quantities' magnitudes or values does not necessarily imply an individual has united those two quantities into images of covariation. Figure 2 involves the co-occurrence of two quantities' variations, but this is conceptually different from covariation as described by Saldanha and Thompson (1998) because each quantity's variation is constrained to its in-the-moment experience or flow. In such a case, that quantity's variation is not held in mind with the "immediate, explicit, and persistent realization that, at every [magnitude], the other quantity also has a [magnitude]" (Saldanha & Thompson, 1998, p. 298). The conceptions captured by Figure 3a and Figure 3b each indicate the quantities' values existing in a multiplicative object. The primary difference between the two, which is illustrated below, is that Figure 3a entails explicit attention to how the quantities' variations occur with respect to specified elapsed time, while Figure 3b has measured durations fade to the background so the relationship is not tied to any particular experience or measured duration.

Illustrating the Framework – A Task

In order to illustrate the framework provided above, I use a task that emerged when constructing second-order models of students' mathematics (Steffe & Thompson, 2000; Thompson, 2008) during a teaching experiment with undergraduate mathematics education students. The teaching experiment explored their reasoning within dynamic situations, including the extent they could construct and re-present relationships between quantities' magnitudes (see Liang and Moore, 2021; Lee et al., 2019; Tasova and Moore, 2020; Moore et al., 2019). The teaching experiment was also part of a larger project focused on capturing middle grades and undergraduate students' covariational reasoning through a series of teaching experiments and conceptual analysis methods (Thompson, 2008). The framework and task, including hypothetical student responses, reflect project findings.

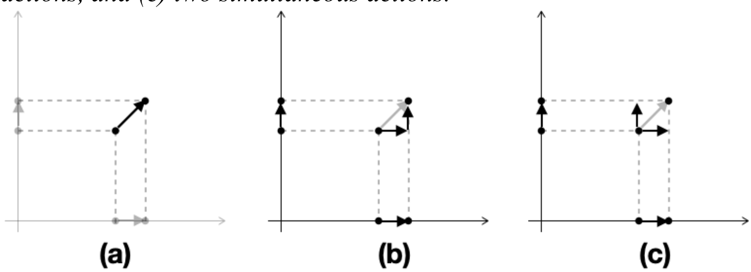
Task Inspirations

With respect to the task below, I drew on three sources of design inspiration beyond those stemming from emergent second-order models of student thinking. As one source, I drew on the tasks and perspectives demonstrated by Saldanha and Thompson (1998) and Carlson et al. (2002). Their tasks primarily included quantities that entail figurative material (e.g., segments) on which to enact quantitative operations (e.g., unitizing and partitioning), and I followed this principle in order to afford students' enactment of quantitative operations. Their tasks also avoided the explicit use of time. Tasks that prompt students to construct graphs with respect to time make it difficult for a researcher to tease out whether the student is reasoning with respect to conceptual or experiential time (Thompson & Carlson, 2017). Further complicating the issue, the quantity of time requires constructing proxy quantities (e.g., segment magnitudes) in order to enact quantitative and covariational operations (e.g., partitioning, unitizing, and iterating). Reflecting these issues, the task below includes two distances (i.e., segment magnitudes that provide the figurative material necessary to enact quantitative and covariational operations) with no reference to elapsed time.

Piaget's (1970) aforementioned work on time provided the second source of inspiration for the task. Piaget described, "It is only by the co-ordination of at least two motions with different velocities that purely temporal relationships can be distinguished from spatial relationships or from intuitive ideas about motion" (p. 26). The task foregrounds relations of simultaneity and succession by prompting the individuals to coordinate two objects in motion, with the two objects varying at different rates with respect to elapsed time. Combining these first two inspirations, I also designed the task to delay providing Cartesian points. I conjectured that providing a point might equally afford students reasoning about the motion of a point as one action (i.e., variation in experiential time), a sequence of actions (i.e., parameterized covariation with elapsed time), or two independent actions that occur simultaneously (i.e., covariation as a multiplicative object) (see Figure 4). But, because a student's observable behavior and utterances can appear the same with each form of reasoning, providing a point makes it difficult to generate the empirical evidence necessary to distinguish between them.

As a third source of inspiration, I drew on the notion of an *abstracted quantitative structure* (Moore et al., 2022). Moore and colleagues introduced the notion of an abstracted quantitative structure in order to provide criteria by which to define the extent to which a student's meaning for a system of quantities is operative. A key criterion of an abstracted quantitative structure is a student's capacity to re-present that structure, which involves the student bringing forth an image of operations that have been previously enacted to either regenerate a previous experience or to accommodate a novel experience via assimilating that experience to operations understood as mathematically equivalent to those enacted during a previous experience. Importantly, students' acts of re-presentation provide insights into aspects of their meanings not readily apparent during prior activity, particularly with respect to covariational reasoning and the role of time (Liang & Moore, 2021; Moore et al., 2022). It is during a student's act of re-presentation that a researcher gains more salient insights into the role of time with respect to their constructed covariational relationship. With respect to the task below, it is their re-presentational activity within a *dynamic geometry environment* (DGE) that enables a researcher to make inferences regarding the role of time in the covariational relationship they constructed prior to the DGE portion of the task.

Figure 4. *The motion of a point as (a) one action, (b) a sequence of actions, and (c) two simultaneous actions.*



The Task: Which One? – Going Around Gainesville (GAG)

"Which One? – GAG" is from a series of tasks titled "Which One?" A "Which One?" task is implemented after an individual constructs, relates, and potentially re-presents a covariational relationship within phenomena or graphical representations (Liang & Moore, 2021). After such actions, a "Which One?" task provides several additional representations of covariational relationships. Examples of these representations include sets of magnitude bars that vary simultaneously or a collection of displayed graphs, sometimes shown as completed and other times shown as in-progress traces. With the representations provided, the individual determines which of the representations, from none to all, accurately capture the relationship they previously constructed (whence the name, "Which One?").

The part preceding "Which One? – GAG" involves a video depicting a car starting in Atlanta and traveling back and forth from Tampa (Figure 5, see Moore et al. (2022) and Moore et al. (2019) for example implementation and data). After viewing the animation, the individual is sequentially asked two graphing tasks (Figure 5). Axes are imposed on Part II. After an individual engages in each part and has constructed what the research team perceives to be a stable understanding of the covariational relationship, they work on "Which One? – GAG."

Which One? – GAG" is a three-part task, with each part consisting of three pairs of magnitude bars presented in a DGE. As support for the reader, videos illustrating each part of the task for each magnitude pair are hosted in a playlist (https://tinyurl.com/249xa7nk). For Part I of the task (see Figure 6a for a snapshot), the individual is presented with three tabs, each containing a pair of magnitude bars. For each pair, one magnitude bar represents the *distance from Atlanta* (*dfA*), and one magnitude bar represents the *distance from Gainesville* (*dfG*). For each pair, the individual can push "Drive" to start or stop the bars dynamically changing together, and the individual can push "Reset" to return the pair to a zero-magnitude *dfA* and corresponding initial *dfG*. The individual is tasked with determining which, if any, of the pairs covary so as to accurately capture the previously determined relationship between the *dfA* and the *dfG*. Table 2 describes the design of each magnitude pair. Pair B and C capture the normative relationship between the two distances.

Figure 5. *The Going Around Gainesville (GAG) task, video at: https://tinyurl.com/2tz4fm9a.*

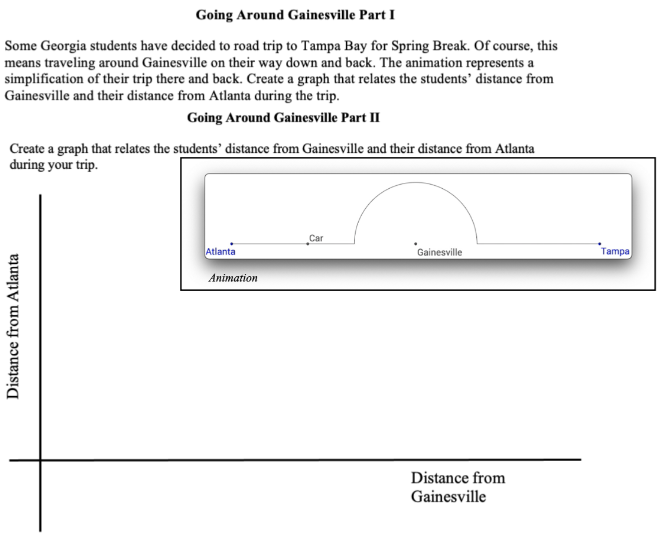


Figure 6. *Example still shots for (a) Pair A – Part I, (b) Pair B – Part II, and (c) Pair C – Part III.*

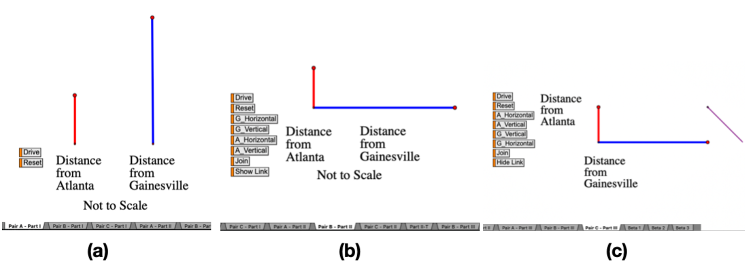
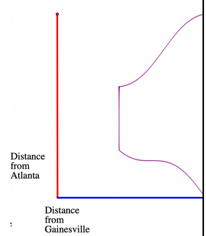
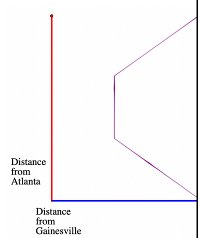
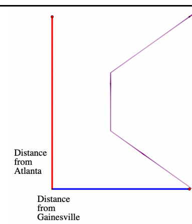


Table 2. *The design of "Which One? – GAG".*

PAIR	RELATIONSHIP DESIGN	PART III GRAPH
A	- With respect to *dfA*: *dfG* decreases at an increasing rate, decreases at a decreasing rate, remains constant, increases at a decreasing rate, and then increases at an increasing rate. - When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a decreasing rate, increases at an increasing rate, increases at a decreasing rate, increases at an increasing rate, and then increases at a decreasing rate; (ii) *dfG* decreases at a constant rate, remains constant, and then increases at a constant rate.	
B	- With respect to *dfA*: *dfG* decreases at a constant rate, remains constant, and increases at a constant rate. - When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a decreasing rate, increases at an increasing rate, increases at a decreasing rate, increases at an increasing rate, and then increases at a decreasing rate; (ii) *dfG* decreases at a decreasing rate, decreases at an increasing rate, remains constant, increases at an increasing rate, and then increases at a decreasing rate.	
C	- With respect to *dfA*: *dfG* decreases at a constant rate, remains constant, and increases at a constant rate. - When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a constant rate; (ii) *dfG* decreases at a constant rate, remains constant, and then increases at a constant rate.	

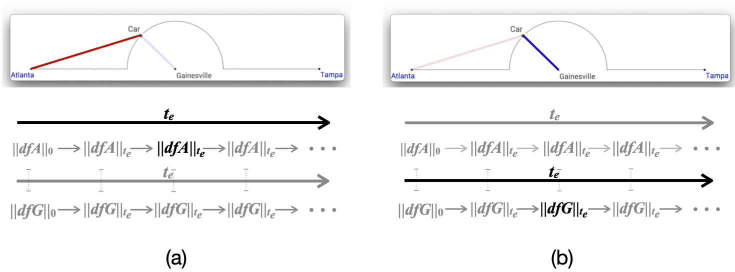
Part II of the task (see Figure 6b for a snapshot) presents the individual the same three pairs of magnitude bars, but they can reorient the magnitude bars, join them, and show a "link" between them. This link represents the process of joining two orthogonal magnitudes to form a Cartesian point. I direct the reader to the playlist (https://tinyurl.com/249xa7nk) to see examples for each pair. An individual is told that each pair matches its respective pair from Part I (e.g., Pair A in Part I, II, and III covary equivalently), and that Part II of the dynamic sketch is designed to help them further explore the extent to which the two magnitude bars capture the determined relationship between the two distances. For Part III of the task (see Figure 6c for a snapshot), the individual is again presented with the same three pairs of magnitude bars. In this case, each pair is oriented orthogonally, a Cartesian point is displayed, and a trace is recorded as the bars move in tandem. Like Part II, the individuals are told each pair matches its respective pair from Part I, and Part III is to aid in further exploring the extent to which the two magnitude bars capture the appropriate relationship between the two distances. During Part II and Part III, an individual is also prompted to reflect on and describe any changes in their assessment of the paired magnitudes, and they can return to the previous parts if desired. They are asked to reflect on difficulties from previous parts and how subsequent parts assist their assessment. As a point of transparency to the reader, I intend Part I to be difficult, both conceptually and in functional design. This is in the hope of eliciting their thinking and allowing spontaneous requests for other representations.

Illustrating the Framework

In this section, I summarize how an individual engaging in each covariation form might conceptualize the DGE. First, focusing on Figure 2 (i.e., $||x||_{t_e} \vee ||y||_{t_e}$), due to the basis in experiential time, an individual reasoning in such a way attends to the variation of each magnitude separately, primarily through the experience of watching the DGE animated continuously. With respect to Pair A, the individual might conclude that *dfG* varies correctly while concluding that *dfA* varies incorrectly. For the former, *dfG* varies appropriately due to its smooth decrease, constancy, and then increase. For the latter, they anticipate that *dfA* should increase at a smooth rate, which reflects the manner in which it increases during the experience of watching the road trip animation. It does not vary in this way (i.e., *dfA* pauses) and thus they discard the pair. With respect to Pair B, and consistent with their response to Pair A, the individual might conclude that *dfG* and *dfA* vary inappropriately due to anticipating both increasing or decreasing at smooth rates, again anticipating that each magnitude reflects the in-the-moment experiences of the variations with the road trip animation. With respect to Pair C, the individual is likely to conclude that both *dfG* and *dfA* vary appropriately due to the smooth variation of each. Across all of the pairs, the individual focuses on each magnitude separately and draws on intuitive or experiential notions of rate to draw their conclusions. Situating Figure 2 with respect to "Which One? – GAG",

Figure 7 captures that each quantity's magnitude is considered separately. It also captures that each is constrained to the flow of continuously watching or imagining the animation, and thus the present state of each quantity (including experiential notions of rate) is always foregrounded. In short, the individual's acts of re-presentation involve the independent variation of each quantity in the flow of experience.

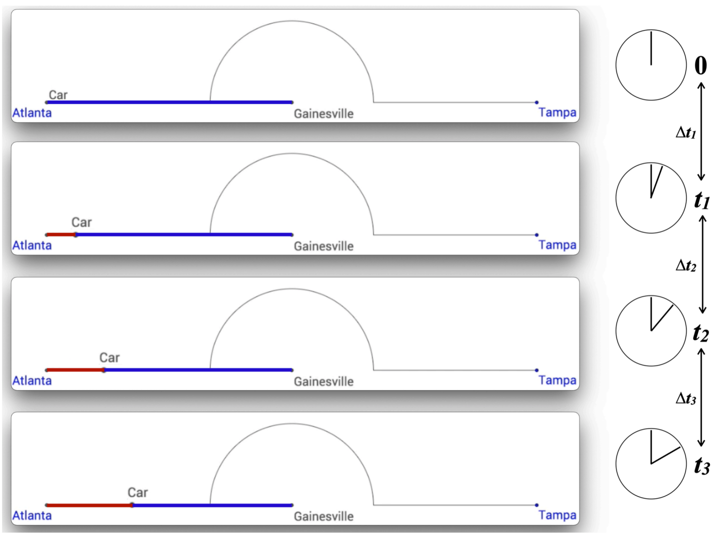
Figure 7. *"Which One? – GAG" and covariation (i.e., (a) for dfA and (b) for dfG) with respect to experiential time.*



For Figure 3a (i.e., $(||x||_t \vee ||y||_t)$), due to the basis in conceptual time, an individual attends to the variation of each magnitude separately, but they coordinate the variation of each using successive durations of elapsed time. This might be accomplished by stepping through states of the DGE and tracking the variation of each quantity with anticipated properties in mind. With respect to Pair A, as the individual tracks through successive, equal duration states of the DGE, the individual might conclude that although *dfG* varies by constant amounts, *dfA* does not vary by constant amounts, and thus, the magnitude bars do not capture the appropriate relationship. With respect to Pair B, the individual might acknowledge the difficulty assessing the pair using the DGE because neither quantity varies at a constant rate with respect to elapsed time. Reflecting that the quantities are cognitively linked through their shared relationship with elapsed time in this form of covariation, the individual might attempt to "Drive" the bars for equal durations of time and then compare the variations of the magnitudes to each other. With respect to Pair C, the individual is likely to conclude that the pair covaries appropriately due to the smooth variation of each, and they might further test this by using successive, equal durations of "Drive." Across all of the pairs, the individual's re-presentation acts involve coordinating each magnitude with equal durations in order to draw comparisons within or across the magnitudes. Because of this, Pair B can lead to a perturbation that stems from the individual anticipating successive equal variations in each quantity for equal variations in duration due to the piecewise linear relationship between *dfG* and *dfA*. Situating Figure 3a with respect to "Which One? – GAG", Figure 8 captures that each quantity's magnitude is coordinated with respect to conceptual time, and their covariation is imagined in terms of moving through specific durations of elapsed time (e.g., the displayed clocks and associated durations). Thus, states of the situation are generated and coordinated by stepping through durations of time as defined by the DGE.

For Figure 3b (i.e., $(||x||_{\#} \wedge ||y||_{\#})$), due to the basis in a disembedded invariant relationship, an individual reasoning in such a way foregrounds coordinating a quantity's variation with respect to the other quantity's variation. Whether Pair A, B, or C, the individual is likely to vary one quantity's magnitude systematically while tracking the variations in the other quantity's magnitude. For instance, the individual might use "Drive" to step *dfA* through successive, equal increases, and then assess the appropriateness of the pair by investigating whether the *dfG* magnitude follows the pattern of constant decrease, constant, and constant increase. An individual engaging in such

Figure 8. *"Which One? – GAG" and covariation with respect to conceptual, elapsed time with equal durations.*



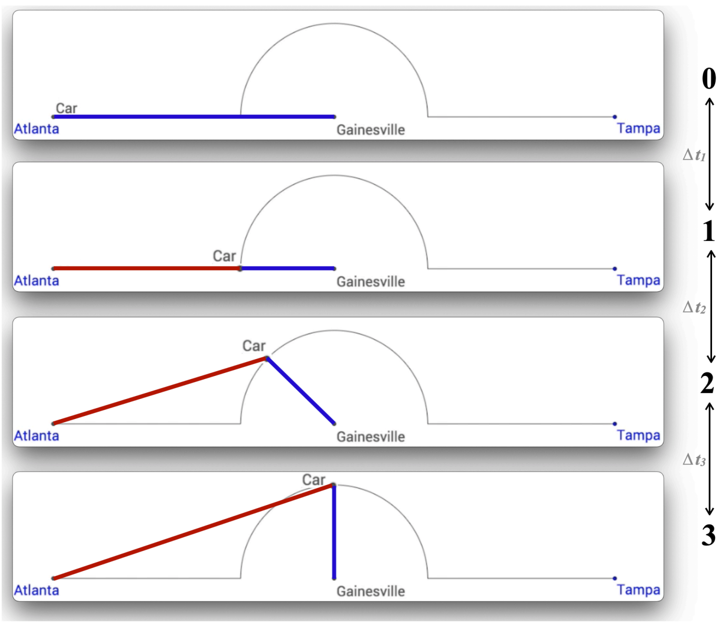
$$\xrightarrow{\quad t_e \quad}$$

$$\|dfA\|_0 \overset{\Delta t_1}{\longleftrightarrow} \|dfA\|_{t_1} \overset{\Delta t_2}{\longleftrightarrow} \|dfA\|_{t_2} \overset{\Delta t_3}{\longleftrightarrow} \|dfA\|_{t_3} \overset{\Delta t_4}{\longleftrightarrow} \cdots$$

$$\|dfG\|_0 \overset{\Delta t_1}{\longleftrightarrow} \|dfG\|_{t_1} \overset{\Delta t_2}{\longleftrightarrow} \|dfG\|_{t_2} \overset{\Delta t_3}{\longleftrightarrow} \|dfG\|_{t_3} \overset{\Delta t_4}{\longleftrightarrow} \cdots$$

covariational reasoning might experience a perturbation stemming from the functionality of the DGE (e.g., it is difficult to use "Drive" to step through equal amounts of increase), but they would not be significantly perturbed by how a single bar varies as the animation plays. They persistently foreground how the bars simultaneously covary, which can lead to expressing annoyance at Part I and motivating a desire for Parts II-III. Situating Figure 3b with respect to "Which One? – GAG", Figure 9 captures that each quantity's magnitude is coordinated with respect to the other quantity, and thus their covariation is imagined strictly in terms of coordinating their magnitudes. Hence, states of the situation are merely constrained by the

Figure 9. *"Which One? – GAG" and covariation with respect to the quantities disembedded from time so that their invariant relationship is taken as the object of thought.*



$$\xrightarrow{\qquad\qquad\qquad t_e \qquad\qquad\qquad}$$

$$\begin{array}{ccccccccc}
\|dfA\|_0 & \overset{\Delta t_1}{\leftrightarrow} & \|dfA\|_1 & \overset{\Delta t_2}{\leftrightarrow} & \|dfA\|_2 & \overset{\Delta t_3}{\leftrightarrow} & \|dfA\|_3 & \overset{\Delta t_4}{\leftrightarrow} & \cdots \\
\mathrm{I}\,o & & \mathrm{I}\,o & & \mathrm{I}\,o & & \mathrm{I}\,o & & \\
\|dfG\|_0 & \overset{\Delta t_1}{\leftrightarrow} & \|dfG\|_1 & \overset{\Delta t_2}{\leftrightarrow} & \|dfG\|_2 & \overset{\Delta t_3}{\leftrightarrow} & \|dfG\|_3 & \overset{\Delta t_4}{\leftrightarrow} & \cdots
\end{array}$$

quantities' covariation, and the individual's re-presentational actions with the DGE pairs center the invariant relationship across the quantities' magnitude states (e.g., in Figure 9, numbered states have replaced the displayed clocks from Figure 8). In fact, the individual might conclude that there are an infinite number of ways that the DGE pairs can vary with respect to elapsed time while achieving the same paired states. The individual would understand that each of those infinite ways re-present the quantities' covariation. For instance, if we took any animation of DGE pairs known to re-present the quantities' covariation, we could replay that animation at any frame rate we desire, including pausing and resuming it in any random manner, and it would still capture the invariant relationship with respect to the magnitude states of *dfA* and *dfG*.

Closing

The framework presented here provides three forms of covariational reasoning that are differentiated based on the role of time and, hence, the extent to which a multiplicative object is formed between the two quantities. Drawing on Piaget's approach to time, the framework positions time as an emergent property of a relationship, as opposed to an attribute in and of itself. That is, time is an emergent property of how the relationship is conceived and, specifically, how the motions of the two objects and their velocities are conceived with respect to each other. Thus, the framework is relevant to any relationship between two quantities' values, including those that implicitly or explicitly involve time as a measured quantity.

The fact that three forms are provided naturally invites questions regarding their developmental or hierarchical nature. Regarding their development, the three forms emerged from work primarily conducted with undergraduate students, and thus I have little insight into their developmental trajectory and relationships. With respect to hierarchy, there is a relative sophistication and generativity across the forms that is reflected in Piaget's exposition of time in combination with Carlson, Castillo-Garsow, Saldanha, and Thompson's descriptions of covariation. This relativeness is captured by Patterson and McGraw (2018), who described,

> We hypothesize that it is advantageous to be able to envision the covariation between two dynamically changing quantities and, to some degree, decouple this image of covariation from a unidirectional, experiential image of the passage of time. This process is essential for developing an understanding of an invariant relationship between two

> quantities and explaining how changes in one variable constrain changes in another variable. (p. 320)

Patterson and McGraw hedge in their hypothesis, as the process of decoupling quantities' covariation from experiential time is intrinsic to the form of covariation captured in Figure 3b and that suggested by Carlson, Castillo-Garsow, Saldanha, and Thompson. Constructing a multiplicative object between two quantities' magnitudes necessarily involves decoupling images of quantities' variations from experiential images of time (Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Furthermore, decoupling quantities' covariation from specified durations of time is necessary for their covariation to be taken as an object of thought so that an invariant relationship can be understood to constrain the two quantities' simultaneous variations (i.e., Figure 3b). I agree with Thompson and Carlson's (2017) assessment that reflecting abstraction (Piaget, 2001) likely provides an explanatory mechanism for such a developmental process of covariation, and future work is needed to identify the specific accommodations and conceptual operations constituting those abstractions, as well as the instructional settings that might engender them.

Although the forms have a hierarchical nature, the implications of such remain unclear. This is particularly true as it relates to how the forms of covariation play a role in students constructing concepts such as rate of change and accumulation. Rate of change involves images of covariation and acts of quantifying an attribute of that covariation (Thompson, 2011). Accordingly, Kertil et al. (2019) introduced the notion of *intensive quantification*. Intensive quantification foregrounds that constructing rate of change involves the quantitative operation of comparing the simultaneous change between covarying quantities so as to determine "how many times as large as" one change is than the other. Implied in this description is that an individual conceives quantities' covariation as a conceptual object that affords constructing an attribute eventually named "rate of change". Supported by empirical findings (e.g., Patterson & McGraw, 2018), the third form of covariation (Figure 3b) is likely an important foundation for a student's construction of such an object and, hence, rate of change as a measure of change. The third form involves constructing a multiplicative object constituted by $||x||$ and $||y||$ so that their relationship is defined precisely and persistently by the simultaneous variation between the two. With that object (i.e., $(||x||_{\#} \wedge ||y||_{\#})$) in mind, students are positioned to quantify rate of change as an attribute of it. The first form (Figure 2) entails both quantities varying with respect to experiential time and is thus subject to intuitive or figurative images of rate. With respect to the second form (Figure 3a), a student is positioned to conceive the rate of change of each quantity's variation with respect to elapsed time. They might then form comparisons between the variation of each quantity with respect to time, but such comparisons are not quantitatively equivalent to constructing rate of change as an attribute of the multiplicative object formed by uniting precisely the two quantities. Although subtle, these differences may have significant implications for their constructing calculus concepts like rate of change and accumulation.

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