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Integration by Substitution: An Emergent Quantitative Reasoning Approach to U-Substitution

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ABSTRACT

In this paper, we present a conceptual analysis for integration by substitution that centers major ideas of quantitative reasoning, including accumulation rates, relationships between measures and unit magnitudes, and the multiplicative dependency of quantities. Our centering of these ideas enables integration by substitution to occur through coordinating accumulation rates and intervals to reconstruct a desired integral structure. Our approach was inspired by the conceptual analysis provided by Jones and Fonbuena (2024), and thus we compare our approach with theirs throughout in order to highlight similarities and differences between the two. We close by acknowledging that a conceptual analysis is only as good as its use in working to support learning and thus call for future work that transitions the analysis to work with students.

Keywords: Conceptual Analysis · Integration by Substitution · Quantitative Reasoning · Accumulation

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Conceptual analysis is a tool for mathematics educators to articulate mathematical ideas with attention to student learning. von Glasersfeld (1995) described conceptual analysis as articulating the ways in which an individual understands a particular mathematical idea, effectively answering the question: "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (p. 78). Thompson (2008) identified two ways conceptual analyses can answer this question. A mathematics educator can answer the question by creating models of how another individual (e.g., a student, teacher, or child) might understand mathematical ideas. Answers of this form are what Steffe and Thompson (2000) termed *second-order models of students' mathematics*, or the *mathematics of students*. Alternatively, a mathematics educator can answer the question by "devis[ing] ways of understanding an idea that, if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise" (Thompson, 2008, p. 45). An answer of this form is of a hypothetical nature as compared to an answer co-constructed with research participants that yields the mathematics of students.

Researchers' use of conceptual analysis spans an impressive developmental range, with topics including rational numbers, arithmetic, function classes, trigonometry, rate of change, and quantitative reasoning (e.g., Confrey & Smith, 1995; Ellis, 2007; Ellis et al., 2015; Johnson, 2015; Jones, 2015; Moore, 2021; Moore et al., 2016; Oehrtman et al., 2008; Paoletti et al., 2023; Simon, 2014; Simon & Glendon, 1994; Steffe & Olive, 2010; Thompson & Carlson, 2017; Thompson & Saldanha, 2003). Specific to the present article, Jones and Fonbuena (2024; 2024) executed a conceptual analysis for accumulation and integration by substitution.[1] Inspired by their analysis, and informed by our empirical and theoretical work in quantitative and covariational reasoning (e.g., Moore, 2021; Moore et al., 2016; Moore et al., 2022), we detail an alternative conceptual analysis for the relationship between accumulation and integration by substitution. The conceptual analysis we provide is an answer of the second way described above; we describe potential meanings for integration by substitution that might be beneficial for students to hold. These meanings have the potential of informing instruction and curricular materials, but our goal here is to only present the conceptual analysis.

In order to situate the topic of our conceptual analysis, we first provided background information on various calculus ideas. We then include a general summary of the approach proposed by Jones and Fonbuena (2024). Aligning with Izsák's (2025) emphasis on the rate of change and measurement perspective on integration by substitution, we next describe quantitative reasoning, magnitude reasoning, and rate of change reasoning to provide foundational ideas for our conceptual analysis. We make explicit connections with Thompson's (1994b) quantitative reasoning and perspective on accumulation. We also contribute to the analysis of integration by substitution by raising the importance of addressing issues of size and also differentiating between the quantities an accumulation rate is *with respect* to and *expressed in terms of*. We subsequently present our conceptual analysis and select comparisons with the Jones and Fonbuena approach. We also close with a broader assessment of the two approaches, as well as a challenge for mathematics educators to pursue for the purpose of constructing a mathematics of students.

[1] We use "integration by substitution" instead of "u-substitution" for a reason we later clarify.

1. Calculus, Accumulation, and Rate of Change

Major calculus ideas have been a significant focus of mathematics education for decades. Calculus concepts like rate of change and accumulation are important not only for mathematics, but also for their application across STEM disciplines. Productive meanings for integration are critical to understanding both mathematical and physical phenomena (Ely & Jones, 2023). More pointedly, using integration meanings flexibly and adaptively in mathematical and physical phenomena requires that those meaning be rooted in quantities and their relationships (Jones, 2015; Jones & Ely, 2023; Thompson, 1994b; Thompson & Silverman, 2007).

Reflecting on the collection of research on integration and its connection to reasoning about quantities and their relationships, Jones and Ely (2023) clarified and summarized two paradigms upon which such work has developed. They termed these paradigms *adding up pieces* and *accumulation from rate* (Jones & Ely, 2023). They described,

> One approach, which we call *accumulation from rate* (AR), focuses on the dynamic covariational relationship between two variables, encapsulated by a rate-of-change function that dictates how variation in one variable leads to continuous accumulation in the other variable…The other approach, which we call *adding up pieces* (AUP), focuses on partitioning a domain object into tiny (infinitesimal) pieces, where each piece has a tiny (infinitesimal) amount of some desired quantity associated with it that are then added up… (Jones & Ely, 2023, p. 192)

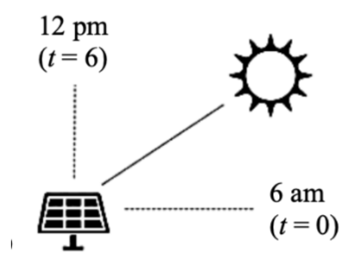
Following these definitions, Jones and Ely (2023) provided an extensive comparison between AR and AUP paradigms. They identified that a primary difference between the two perspectives is that the AR paradigm rests on the intrinsic relationship between derivatives and integrals (i.e., rate of change and accumulation), while the AUP paradigm "places priority on understanding the definite integral" (Jones & Ely, 2023, p. 129). By leaving implicit the cognitive actions associated with rate reasoning including partitioning and disembedding (see Steffe & Olive, 2010), the authors illustrated that the AUP perspective enables symbolic representations of integration to emerge as capturing the direct summation of target quantity pieces. The AR perspective, on the other hand, involves making direct use of rate structures and their multiplicative nature to construct accumulation in one quantity with respect to another.

For our analysis, the distinctions between the AUP and AR paradigms are not critical because of integration by substitution's intrinsic connection to the chain rule. The target quantity in integration by substitution contexts has the underlying structure of a multiplicative relationship between an accumulation rate (e.g., $f'(g(t))g'(t)$) and an interval (or partition) over which the target quantity accumulates at that rate (e.g., dt). We make and draw on this intrinsic connection throughout this article.

2. Summarizing the Jones and Fonbuena (2024) Approach to Integration by Substitution

We summarize Jones and Fonbuena's (2024) conceptual analysis here for the reader's purpose. We also point the reader to Jones and Fonbuena's other works (Fonbuena & Jones, 2024; Jones, 2022; Jones & Ely, 2023) for more detailed descriptions of important, foundational ideas to their conceptual analysis, as well as illustrations of it. Jones and Fonbuena's (2024) example context was that of a solar panel generating energy over a 6-hour period, in which elapsed time, produced energy, and power are each a relevant quantity (Figure 1). The authors chose the units of hours (hr), kilojoules (kJ), and kilojoules per hour (kJ/hr) for each quantity, respectively. The authors defined the relationship between power and elapsed time as $P(t) = 1500\sin\left(\frac{\pi}{12}t\right)$. They then defined $dE = 1500\sin\left(\frac{\pi}{12}t\right)dt$ to be the power accumulated over an essentially infinitesimal time interval, *dt*. Thus, the total energy accumulated from 0 hours to 6 hours elapsed since 6:00 AM is $E = \int_0^6 1500\sin\left(\frac{\pi}{12}t\right)dt$.

Figure 1. The solar panel context (Jones & Fonbuena, 2024, p. 915).



Against the backdrop of this context and integral, Jones and Fonbuena (2024) motivated integration by substitution using the question: "What if we wanted to track the total energy in terms of the *angle* the sun makes with the horizon (in radians), rather than the time on the clock? This is the fundamental quantitative question we claim integration by substitution is based on: How do you convert from one main *input quantity* to a different main *input quantity*?" (p. 915). Having framed integration by substitution as a conversion process of input quantities, the authors provided an explanation for how the conversion influences each part of their integration structure. For the partition (or differential), they defined the conversion formula $d\theta = \frac{\pi}{12}dt$ (or equivalently $dt = \frac{12}{\pi}d\theta$). They then identified the need to define the integrand quantity in terms of angle (measure), and they relied on the *nested relationship* (Jones, 2022)[2] between time, angle, and power to rewrite $P(t) = 1500\sin\left(\frac{\pi}{12}t\right)$ as $P(\theta) = 1500\sin(\theta)$.[3] With the partition and integrand quantity handled, they identified that the integral is to be defined in terms of angle (measure) rather than time. Defining the bounds of the integral by using the appropriate quantity's measures, they obtained $\int_0^{\pi/2} 1500\sin(\theta)\frac{12}{\pi}d\theta$. The authors appealed to unit-tracking throughout their process to illustrate and justify each step as it relates to their desired result.

Generalizing from this example, the authors defined *input quantity*, *target quantity*, and *integrand quantity* as presented in Table 1. They summarized,

> With this terminology in place, recall that a definite integral answers the following question: If a target quantity is some combination of an input quantity and an integrand quantity, how can we determine its total amount if the integrand quantity varies over the input quantity? Relatedly, we claim that u-

[2] Jones (2022) describes this as a *nested multivariation (MV) structure*. For the purposes of this analysis, it is sufficient to understand a nested MV relationship as that of basic function composition in which quantities are in a "chain of influence". In this case, time, angle, and power form the chain.

[3] We reproduce the authors' notation despite potential inconsistencies under some function notation conventions. In this case, the function name *P* is used to define two distinct real-valued functions.

> substitution now answers this question: How can we determine the total amount of the target quantity if we want to switch from tracking it in terms of the original input quantity to a new, different input quantity? This conversion is based on an inherent nested MV structure going from *original input → new input → integrand quantity*. In our solar panel example, the nested MV was *time → angle → power*, described by the function composition $P(\theta(t))$. (Jones & Fonbuena, 2024, p. 916)

To connect to a three-part integration structure, they summarized the integration by substitution process as presented in Table 2. Table 2 underscores that the authors' approach is one in which the measure of the partition interval, the integrand quantity, and the bounds of the integral (or "sum") are re-written in terms of the new input quantity via its composition relationship with the original input quantity. Jones and Fonbuena (2024) stated this process can be completed in any order.

Table 1. Some terminology from Jones and Fonbuena (2024, p. 916).

Term	**Definition**
Input Quantity	"…the quantity that the other two depend on and that gets partitioned into infinitesimal pieces (*time*, in our example)."
Target Quantity	"…the quantity whose bits are added up by the integral and whose total amount is given by the value of the integral (*energy*, in our example)."
Integrand Quantity	"…the quantity that functionally depends on the input quantity and that also combines with the input quantity to produce the target quantity. In our example, the integrand quantity was *power*."

Table 2. The resulting AUP three-part structure and process for Jones and Fonbuena (2024, p. 917), reproduced verbatim.

Initial AUP parts	**Corresponding parts in the u-substitution structure**	
1. Partition: da	1. Differential: Convert partition from da pieces to equivalently-sized pieces in db.	$da \rightarrow [conversion]db$
2. Target quantity: $Q \cdot da$	2. Integrand: Convert integrand quantity, Q, from depending on a to depending on b. Target quantity now determined entirely by new input.	$Q(a) \rightarrow Q(b)$ $Q(a)da \rightarrow Q(b)[conv]db$
3. Sum: $\int_{a1}^{a2} Q(a)da$	3. Bounds. Convert sum from running across original input pieces to new input pieces.	$a_1 \leq a \leq a_2 \rightarrow$ $b_1 \leq b \leq b_2$

3. Foundational Ideas for Our Approach to Integration by Substitution

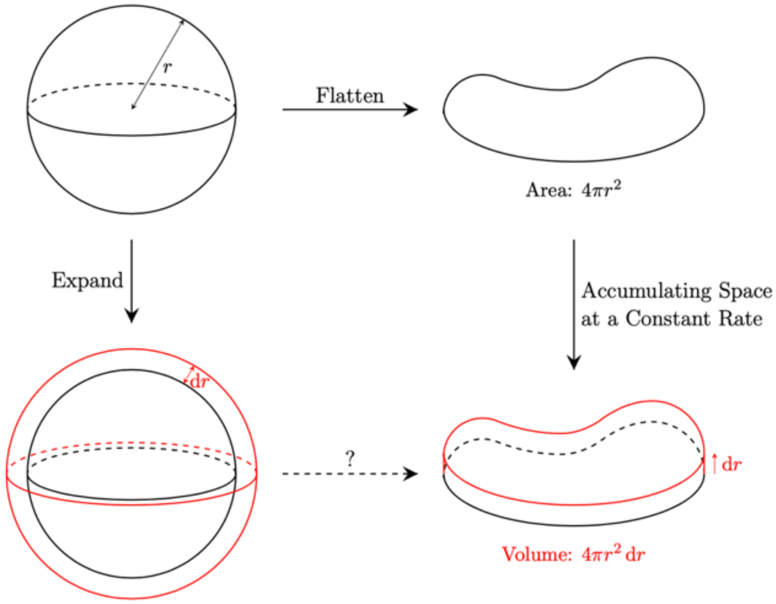
Our conceptual analysis is built on our understanding and use of quantitative and covariational reasoning theories. A growing body of literature has illustrated quantitative and covariational reasoning's importance for students' mathematical development and teachers' capacity to target coherent mathematics in their instruction (e.g., Karagöz Akar et al., 2022; Steffe et al., 2014; Steffe & Olive, 2010). Thompson's theory of quantitative reasoning approaches mathematical reasoning as a process of conceiving phenomena in terms of measurable attributes—called quantities—and relationships between these attributes—called quantitative relationships (Smith III & Thompson, 2007; Thompson, 1989, 2011).

A quantity is a cognitive object that results from the construction of an attribute, a measurement process by which to quantify that attribute, and a measure that conveys a relationship between the amount-ness of a quantity and a unit magnitude (Thompson, 2011). A key feature of quantitative reasoning is that it extends beyond values and calculational conversions to center the quantitative operations between measures, unit magnitudes, and, in our case, accumulation.

Covariational reasoning is a form of quantitative reasoning that involves the "cognitive activities involved in coordinating two [or more] varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Covariational reasoning's relevance for calculus is somewhat self-evident given that calculus is the *mathematics of change*, and researchers have provided extensive empirical backing to the link between calculus ideas and covariational reasoning (Byerley, 2019; Carlson et al., 2002; Carlson & Oehrtman, 2004; Ellis et al., 2020; Ely & Jones, 2023; Engelke, 2007; Oehrtman, 2008; Oehrtman et al., 2008; Thompson et al., 2013; Thompson & Carlson, 2017; Thompson & Silverman, 2007). Most relevant to the work here, Thompson (1994b) outlined an approach to the Fundamental Theorem of Calculus that emphasizes the accumulation of quantities and their rate of change are inseparable. Accumulation and rate of change are two sides of the same coin, with accumulation and the rate of change of quantities necessarily involving each other (Thompson, 1994b). We capture this aspect with an analysis of a context used by Jones and Fonbuena (2024).

In their analysis, they present a sphere whose radius is increasing with respect to time. They described that the accumulation of volume for an infinitesimal *dt* is determined by using the "product of the [sphere's] surface area and the corresponding change of radius" (Jones & Fonbuena, 2024, p. 918), thus not directly presenting the accumulation of volume in terms of a rate. We view volume growth as quantitatively more complex than an arithmetic operations between values (Brady & Lehrer, 2021; Lehrer & Schauble, 2023; Simon, 1995); quantitatively understanding a sphere's volume growth goes beyond understanding that it is numerically equivalent to the calculational product of a sphere's surface area and another dimension (or interval of time; Lehrer & Schauble, 2023; Thompson, 1994b). A quantitative understanding of volume and its growth involves conceiving a sphere's surface area as providing the material constraint for the instantaneous accumulation rate of volume with respect to a particular length (Thompson, 1994b). Such a conception enables one to quantitatively justify why the product of a surface area value and a length yields a volume; volume growth is generated by envisioning an object accumulating volume at a constant rate numerically equivalent to a surface area value (Figure 2). Volume growth is quantitatively formed by the relationship between the volume accumulation rate (in cubic length units per length unit) and the quantity volume is accumulating with respect to (in a length unit). More generally, our sensitivity to the quantitative structure of volume growth reflects Thompson's primary purpose of quantitative reasoning (Smith III & Thompson, 2007; Thompson, 1989, 2011), which is to clarify quantitative aspects and nuances of mathematical concepts in ways that go beyond numerical relationships and known calculations or formulas. Said another way, Thompson's introduction of quantitative reasoning was not for the purpose of attaching meaning to known facts and symbolic representations or techniques, but instead to re-envision the mathematics we might strive to engender working with our students so that it emerges from reasoning about quantities and their relationships (Thompson, 2012, 2013).

Figure 2. Illustrating the relationship between the surface area of a sphere, the material constraints for volume growth, and volume accumulation rate.



3.1 Defining Several Quantities and Related Phrases

Drawing on Thompson's work, our conceptual analysis makes transparent the role of variation, rate of change, and accumulation in integration by substitution. To support this transparency, we use Table 3 to define the quantitative attributes: accumulation, accumulation interval, accumulation rate, accrual, and accrual interval. We also introduce the terms *in terms of* and *with respect to* as a tool to aid the linguistic clarity of our conceptual analysis.

Table 3. Some terminology adopted from Thompson (1994b) and two additionally defined phrases.

Term	**Definition**
Accumulation	The total amount a quantity (i.e., Q_1), has changed from some reference amount.
Accumulation Interval	The interval of a second quantity's (i.e., Q_2) magnitude or measure over which another quantity (i.e., Q_1) accumulates.
Accumulation Rate	The rate at which a quantity (i.e., Q_1) accumulates with respect to changes in a quantity defining its accumulation interval (i.e., Q_2).
Accrual	The sub-accumulations of a quantity that constitute the total accumulation of that quantity. Any total accumulation can be conceived via the sum of accruals. For instance, the accumulation of Q_1 occurs via accruals of Q_1
Accrual Interval	The interval of a quantity's magnitude or measure over which a second quantity accrues. For instance, accruals in Q_1 occur over accrual intervals of Q_2.

Term	Definition
…in terms of…	A phrase to define functional dependency. To say that quantity Q_1 is (expressed) in terms of quantity Q_2 is to make explicit we are defining a function in which quantity Q_1 is expressed as a function of quantity Q_2. In the presence of variables, we say that $f(x)$ is such that f is evaluated at x, and the output $f(x)$ is expressed in terms of the input x.
…with respect to…	A phrase to define covariational and multiplicative dependency. To say that quantity Q_1's rate is with respect to quantity Q_2 is to make explicit we are defining quantity Q_1's change multiplicatively relative to quantity Q_2's change. To say that we are defining quantity Q_1's accumulation rate with respect to quantity Q_2 is to say that the expression defining quantity Q_1's accumulation rate conveys the multiplicative relationship of quantity Q_1's accrual relative to quantity Q_2's simultaneous accrual.

Accumulation (or accumulated amount) refers to the total amount a quantity, Q_1, has changed (from some reference amount, which may or may not have measure 0). A quantity's accumulation is often conceived to occur over an interval of at least one other quantity, Q_2, and the accumulation of Q_1 is said to occur over a Q_2-*accumulation interval*. The accumulation interval can be fixed, varying, or arbitrary. A fixed accumulation interval—indicated by specified interval bounds such as from $t = 1$ to $t = 2.5$—defines the accumulated amount of Q_1 as the difference between the total amount at the end of the interval and that at the beginning. An accumulation interval effectively establishes the amount of Q_1 at the beginning of the interval as the reference amount. *Accumulation rate* is the rate at which the accumulating quantity changes with respect to another quantity. This other quantity may or may not be the other quantity the accumulation is considered to occur over and an interval of.

We draw attention to the difference between an accumulation rate being *with respect to* or *expressed in terms of* some other quantity. When we write that an accumulation rate is with respect to another quantity, Q_2, we are making a statement about the multiplicative relationship between the accumulations of Q_1 and Q_2; Q_1 varies by some amount per change in Q_2. When we say that an accumulation rate is expressed in terms of another quantity, Q_2, we are making a statement of functional dependency; the accumulation rate for Q_1 depends upon or is defined by Q_2. Returning to the Jones and Fonbuena (2024) example above, writing power as $1500\sin\left(\frac{\pi}{12}t\right)$ is to define an accumulation rate *with respect to* and *expressed in terms of* time measured in hours. The rate is of the structure kJ/hr, and it is expressed so that it is defined by t. Writing power as $1500\sin(\theta)$ is to define an accumulate rate *with respect to* time measured in hours and *expressed in terms of* angle measure in radians. The rate is of the structure kJ/hr and expressed so that it is defined by θ. The two accumulation rates are equivalent in their multiplicative structure, but differ in their written functional dependency.

A total accumulation can be envisioned as occurring through sub-accumulations, called *accruals*; accruals are the quantities "by which the accumulations are constructed" (Thompson, 1994b, p. 5). Accruals are a critical conception for calculus because they are the quantity at the heart of envisioning a target quantity in terms of infinitesimal variations occurring over the interval of another quantity (Thompson, 1994b). We use *accrual interval* to refer to the interval of Q_2 over which an accrual of Q_1 occurs. Just as the terms accumulation and accumulation interval differentiate between a total quantity and the other quantity over which it is accumulating, the terms accrual and accrual interval enable us to do the same with respect to Q_1 and Q_2 accruals. Thompson's figure (Figure 3) illustrates this relationship in the context of a quantity accumulating at a constant rate of change with respect to another quantity. Alternatively, we can consider the accumulation of energy over some accumulation interval of elapsed time to occur in accruals over specified accrual intervals (Figure 4). In this case, energy accumulates at a non-constant rate of change with respect to time that is consistent with the context provided by Jones and Fonbuena (2024).

Figure 3. Thompson's (1994b, p. 233) illustration of a quantity accumulating at a constant rate of change with respect to another quantity through coordinated accruals.

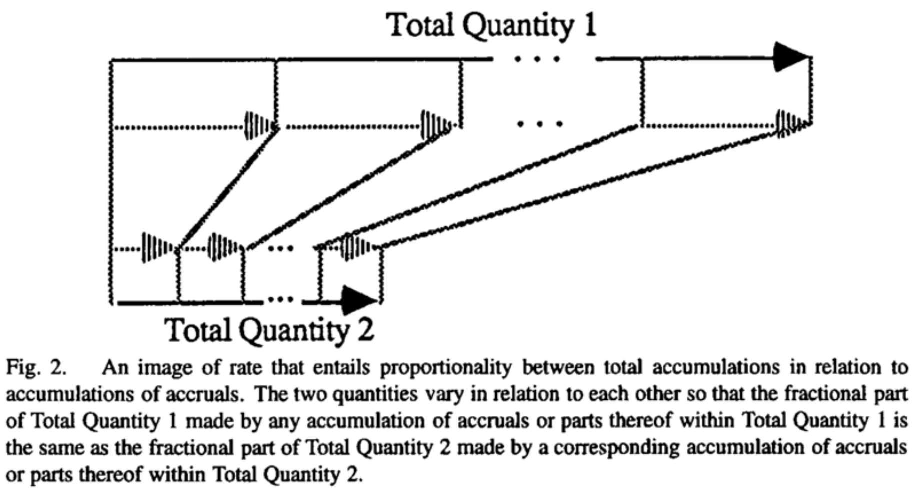
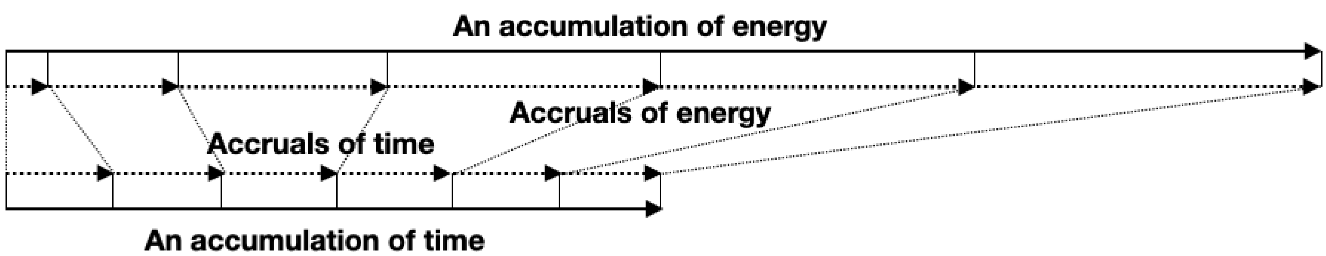


Figure 4. Energy accumulating at a non-constant rate with respect to time via simultaneous accruals.



To connect our terms (Table 3) and those introduced by Jones and Fonbuena (2024) (see Table 1 and Table 2), we provide Table 4. Jones and Fonbuena (2024) used target quantity to encompass both accumulation and accruals. Target quantity refers to a "total amount" given by the "sum" or integral, which is an accumulation, as well as the accruals that are "added up" by the integral. They used input quantity to define the quantity that the target quantity's accumulation is dependent on, and they used it to define the accumulation interval. They described that a target quantity's accumulation can be conceived as dependent on several input quantities (e.g., the accumulation of energy as dependent upon time elapsed or angle rotated), and their use of *dependent on* is consistent with our use of *expressed in terms of*. Their use of integrand quantity aligns with our use of accumulation rate, but their approach is not entirely equivalent. They operated from the assumption that a target quantity is a combination of an integrand quantity and input quantity. We give explicit attention to the multiplicative structure of an accumulation rate. Whereas Jones and Fonbuena (2024) focused on the input quantity solely in terms of dependence, we make explicit use of both the *with respect to* and *expressed in terms of* aspects of accumulation and accumulation rates. Their use of partition or differential aligns with our use of accrual interval. Their use of bounds aligns with our use of accumulation interval.

Table 4. Relating terms from our conceptual analysis and that of Jones and Fonbuena (2024).

Our Term	**Related Jones and Fonbuena (2024) Term**
Accumulation	Target Quantity
Accumulation Interval	Bounds
Accumulation Rate	Integrand Quantity
Accrual	Target Quantity
Accrual Interval	Partition or Differential
…in terms of…	…dependent on…
…with respect to…	No related term

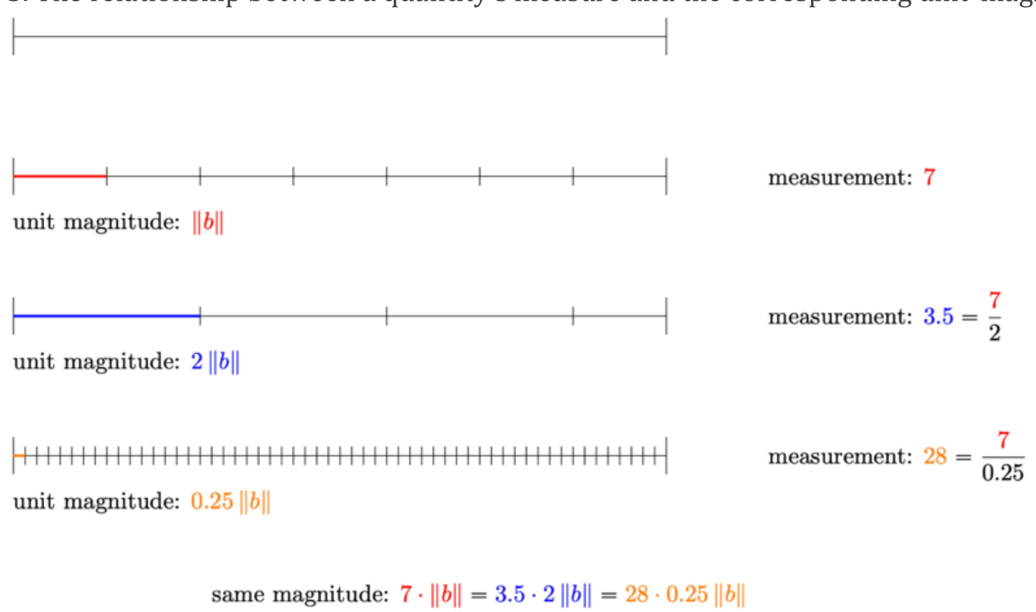
3.2 Instantaneous Rate of Change

Expanding on the covariational structure of an accumulation rate, our conceptual analysis relies on a meaning for instantaneous rate of change as a hypothetical constant rate of change (Thompson, 1994b). To claim Q_1 has an instantaneous rate of change of c with respect to Q_2 at a Q_2-value of a means that if Q_1 were to vary at a constant rate of change with respect to variations in Q_2 from that value a, that constant rate of change would be c. The meaning of a car having an instantaneous speed of 65 miles per hour 2.5 hours into their trip is that, if that car were to travel at a constant speed from that moment in the trip forward, it would maintain a constant speed of 65 miles per hour. Furthermore, if Q_1 accumulates at a constant rate of change of c with respect to Q_2, then each quantity's accumulation occurs through accruals in a proportional relationship (and hence, their accumulations exist in a proportional relationship; see Figure 3). This meaning for instantaneous rate of change is critical for interpreting and changing the *with respect to* quantity of an accumulation rate. To illustrate, consider changing the accumulation rate of a car's distance traveled so that it is with respect to an accrual of time in minutes rather than hours. The unit minute is 1/60 times as large as the unit hour, and thus the accrual of distance in a unit minute is 1/60 times as large as that occurring in a unit hour. It follows that the accumulation rate with respect to a time measure in minutes is 65/60 miles per minute at 2.5 hours into the trip. The car's instantaneous speed measurement with respect to the unit minute is 1/60 times as large as the instantaneous speed measurement with respect to the unit hour.

3.3 Measures and Magnitudes

The previous paragraph's approach changing the *with respect to* aspect of an accumulation rate lies within coordinating magnitudes, units of measure (or unit magnitudes), and measures. Providing clarity to researchers' and educators' typically ambiguous use of *size*, Thompson et al. (2014) distinguished between different ways to quantify size while emphasizing that a quantity's size is an invariant amountness independent of specific units. They explained a sophisticated image of quantity involves understanding the amount (or magnitude) of a measured quantity as invariant across different units used to produce the measured amounts (Byerley & Thompson, 2017; Thompson et al., 2014). Said another way, the magnitude of a quantity is the invariant amount conveyed by all unit-measure pairs (Moore et al., 2016; Thompson, 2011). A person reasoning about a quantity's size in such a way is persistently mindful of the multiplicative relationship between a measure and its unit magnitude. A measure of a in unit magnitude $||b||$ means the quantity has a magnitude a times as large as $||b||$ (Figure 5). That person is also persistently mindful of the inversely proportional relationship between a measure and the size of the unit magnitude (Izsák, 2025; Thompson et al., 2014). If one is to change the size of the unit magnitude by a factor of c, the resulting measure is $1/c$ times as large as the measure in the previous unit magnitude (Figure 5). In summary, a sophisticated image of a quantity's amount (and measure) involves understanding and anticipating how variations in a unit magnitude results in variations to a quantity's measure so as to maintain the quantity's invariant magnitude.[4]

Figure 5. The relationship between a quantity's measure and the corresponding unit-magnitude.

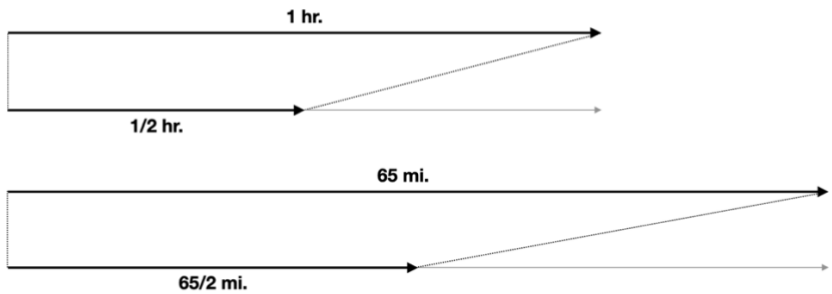


Returning to the speed example above, imagine desiring representing speed in a time unit magnitude—the half-hour—1/2 times as large as the given time unit magnitude—the hour. We can draw on this unit magnitude relationship to conclude the accrued distance for the new time unit magnitude will be ½ times as large as that for the original time unit magnitude (Figure 6), yielding a speed measure of 65/2 miles per half-hour. Importantly, during this process, we hold in mind that the quantity being determined—constant speed as a proportional relationship between accumulations of distance and elapsed time—is an accumulation rate that is remaining invariant in its amount but being expressed in different units; 65/2 miles per half-hour is of the same magnitude as 65 miles per hour, although the latter's value is

[4] See Izsák (2025) for a description of multiplication as coordinated measurement that provides one way to conceive this relationship.

numerically greater than the former. We use this example to illustrate that "size" is an ambiguous quantifier unless one is explicit about whether one is comparing magnitudes or the numbers representing quantities' measures.[5]

Figure 6. The distance accumulated per different unit time magnitudes for a constant speed.



In summary, our approach is built on theories of quantitative and covariational reasoning that make explicit measurable attributes of integration and how they are related in the context of integration by substitution. These quantities include an accumulation quantity and its corresponding accruals, another quantity defining an accumulation interval and its corresponding accrual intervals, and an accumulation rate that as a relative measure of two quantities' accruals. We give explicit attention to how a quantity is expressed in terms of another quantity, as well as how a particular quantity's accumulation or accrual is measured (multiplicatively) with respect to another quantity's accumulation or accrual. The latter involves coordinating accrual interval sizes with quantities' measures and the relative size of unit magnitudes.

4. A Few Words about Quantitative Reasoning and the Chain Rule

Jones and Fonbuena (2024) used nested relationships to substitute expressions by using expressions relating quantities' measures. Building on this, we identify a quantitative understanding of nested rate relationships that centers both quantities' measures and relationships between unit magnitudes and accruals. We also use the relationship between a function's output, its argument, and its input in the context of using rate relationships to determine accumulation rates with respect to different unit magnitudes. This exploration is at the heart of the chain rule, to which integration by substitution is intrinsically tied.

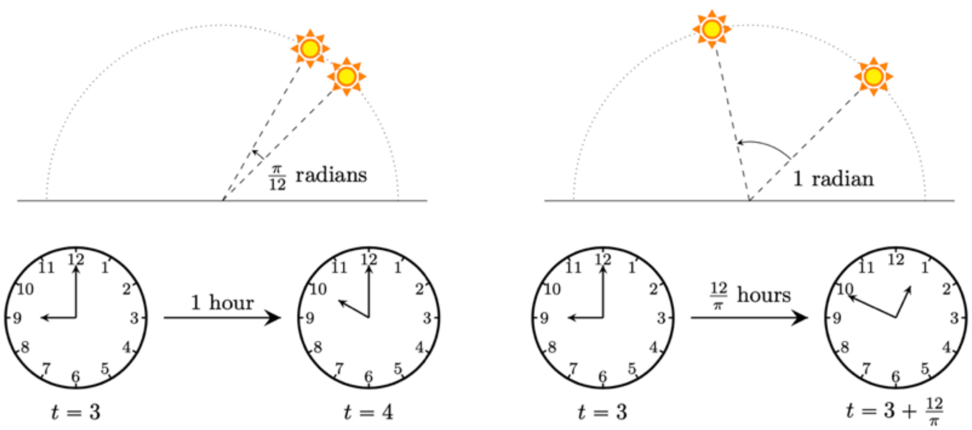
In integration by substitution, the accumulation function has the form $h(x) = f(g(x))$ and the accumulation rate has the form $h'(x) = f'\big(g(x)\big)g'(x)$. Using the energy context to illustrate a quantitative understanding of *the chain rule*, consider $f(\theta) = \frac{18000}{\pi} - \frac{18000}{\pi}\cos(\theta)$, $g(t) = \frac{\pi}{12}t$, and $\theta = g(t)$. The function output $f(\theta)$ represents the accumulated energy in kJ evaluated at θ radians and $g(t)$ represents the angle accumulated in radians evaluated at t hours. The chain rule involves determining the energy

[5] The importance of this is captured by the question, "What is bigger, 60 minutes or 1 hour?" The answer is entirely dependent on the meaning for "size" one has in mind.

accumulation rate in terms of *and* with respect to t, as opposed to θ. A quantitative interpretation of the chain rule proceeds in the following way:

1. Desiring the accumulation rate in terms of t and with respect to t, we first represent the energy accumulation in kJ in terms of t hours, $h(t) = f(g(t))$.
2. The accumulation rate of angle measure in terms of and with respect to t is $g'(t) = \frac{\pi}{12}$, or $\pi/12$ radians per hour. This means that an angle measure accrual interval that is $\pi/12$ times as large as a unit radian occurs during a unit hour accrual interval. Alternatively, per unit radian accrual, time accrues by $12/\pi$ hours (Figure 7).
3. The energy accumulation rate in terms of and with respect to θ is $f'(\theta) = \frac{18000}{\pi}\sin(\theta)$. Desiring the accumulation rate expressed in terms of t, we can write the energy accumulation rate with respect to θ as $f'\big(g(t)\big) = \frac{18000}{\pi}\sin\left(\frac{\pi}{12}t\right)$.
4. Using 2 and 3 above, if the accumulation rate expressed in terms of t and with respect to θ is $f'\big(g(t)\big) = \frac{18000}{\pi}\sin\left(\frac{\pi}{12}t\right)$, then the accumulation rate expressed in terms of and with respect to t is $f'\big(g(t)\big)g'(t) = \frac{18000}{\pi}\sin\left(\frac{\pi}{12}t\right)\frac{\pi}{12}$, or $f'\big(g(t)\big)g'(t) = 1500\sin\left(\frac{\pi}{12}t\right)$.

Figure 7. Relating unit accruals between time and angle measure.



Generically, if E accumulates at a rate of $f'(\theta)$ with $\theta = g(t)$, then E accumulates at a rate of $f'(g(t))$ with respect to θ and θ accumulates at a rate of $g'(t)$ with respect to t. An accrued radian angle measure is $g'(t)$ times the accrued time measure in hours, and the accrued energy measure in kJ over that accrual interval is $f'(g(t))$ times the accrued radian angle measure. It follows that E accumulates at a rate of $f'(g(t))g'(t)$ with respect to and expressed in terms of t. Here, the scaling of $f'(g(t))$ by $g'(t)$ yields the energy accumulation rate with respect to t by multiplicatively chaining the E accumulation rate with respect to θ and the θ accumulation rate with respect to t; the conversion emerges from the need to construct the accumulation rate with respect to the desired structure. Written formally, if $h(t) = f(g(t))$, then $h'(t) = f'(g(t))g'(t)$. The latter forms the integrand quantity in integration by substitution, and the production of this integrand quantity illustrates its roots in accumulate rates.

5. Our Alternative Approach to Integration by Substitution

We begin by using the solar panel context above. Energy accumulates at a rate of $1500\sin\left(\frac{\pi}{12}t\right)$ kJ/hr (with respect to and in terms of t), and our interest is how much energy, E, accumulates from 0 hours to 6 hours. We have $E = \int_0^6 1500\sin\left(\frac{\pi}{12}t\right)dt$. Following Jones and Fonbuena (2024), we ask: "What if we wanted to track the total energy in terms of the *angle* the sun makes with the horizon (in radians), rather than the time on the clock?"[6] (p. 915). In our approach, there are two potential components to pursuing this question. Firstly, we can desire to *express* the accumulation *in terms of* θ, as opposed to t. Secondly, we can desire to *determine* the accumulated energy using the energy accumulation rate *with respect to* θ, as opposed to t. We simultaneously give attention to each by incorporating Thompson and colleagues' (2014) ideas of quantity and unit-measure pairs discussed above.

First, we determine the quantitative relationship between angle measure and elapsed time. We use that relationship to not only change the quantity that other quantities are expressed in terms of, but to also determine the accumulation rate with respect to θ. For the context under consideration, we know each of the following to be true:

1. A unit hour accrual of time corresponds to a π/12 radian accrual. Thus, a unit radian accrual of angle measure is 12/π times as large as that which accrues for a unit accrual of time (i.e., relating accrual interval magnitudes for a unit radian and a unit hour, Figure 7).
2. An accumulated angle measure in radians, θ, is π/12 times as large as the accumulated time measure in hours, t (i.e., relating the measures θ and t, or $\theta = \frac{\pi}{12}t$).
3. For any number of hours elapsed accrued, the number or radians accrued is π/12 times as large (i.e., relating the measures $d\theta$ and dt, or $d\theta = \frac{\pi}{12}dt$).

Using 1 above and the assumed meaning for instantaneous rate of change, per unit radian we accrue 12/π times the energy accrued per unit hour.[7] If we accumulate $1500\sin\left(\frac{\pi}{12}t\right)$ kJ per unit interval of time at t hours, we accumulate $\left(\frac{12}{\pi}\right)1500\sin\left(\frac{\pi}{12}t\right)$ kJ per unit interval of angle measure at t hours. The accumulation rate is now determined *with respect to* θ. We express that rate *in terms of* θ using 2., obtaining $\left(\frac{12}{\pi}\right)1500\sin(\theta)$ kJ per unit interval of angle measure at θ radians. We have expressed the energy accumulation rate *with respect to* and *in terms of* θ. We consider energy to accrue over accrual intervals of $d\theta$ and a total accumulation interval of 0 to π/2, obtaining a total accumulated energy of $E = \int_0^{\pi/2}\left(\frac{12}{\pi}\right)1500\sin(\theta)d\theta$. Our alternative process is a reconstruction of the accumulation rate, accrual interval, and accumulation interval using the given accumulation rate and the relationships between accruals' measures and magnitudes. Our process is not one of defining *u du* formulas, substituting, and regrouping. The accumulation rate (i.e., integrand quantity) is persistently conceived in relation to its constituent unit-measure pairs, while ensuring the other aspects of the integral are expressed in ways quantitatively commensurate with it. Said another way, the integration expression emerges to satisfy the

[6] In our closing discussion, we allude to the importance of creating an intellectual need for asking and pursuing this question.
[7] The assumed meaning for instantaneous rate of change is used for any determination of an accumulation rate from here forward.

goal of representing the total accumulation in terms of and with respect to the desired quantity, and it is accomplished by considering how our change in accumulation rate structure requires a scaled measure. Our choice to use the phrase *integration by substitution* in place of *u-substitution* reflects the fact that the entire integral $\int_0^{\pi/2}\left(\frac{12}{\pi}\right)1500\sin(\theta)d\theta$ emerges as a substitute representation for the original integral. Regardless, each total accumulation integral is understood as capturing the same accumulated quantity, and each is derived by constructing its parts so that it is expressed in terms of and with respect to a desired quantity.

5.1 Tracking the With Respect to and In Terms Of compared to Unit-Tracking

Key to our approach is giving explicit attention to the *with respect to* and *in terms of* components of the quantities involved in integration by substitution. Using integral notation with associated units to capture the with respect to and subscripts to identify what the parenthetical quantity is expressed in terms of, we started with

$$\int_{\mathrm{hr}}^{\mathrm{hr}}\left(\frac{\mathrm{kJ}}{\mathrm{hr}}\right)_t\ \mathrm{hr}$$

and then we re-constructed the total energy as

$$\int_{\mathrm{rad}}^{\mathrm{rad}}\left(\frac{\mathrm{kJ}}{\mathrm{rad}}\right)_\theta\ \mathrm{rad}.$$

This was accomplished by leveraging the relationships between quantities including unit-measure pairs to rewrite the accumulation rate so that it was with respect to and expressed in terms of the desired quantity and unit. We then wrote the differential and bounds of integration to be consistent with that accumulation rate and dependency structure.

We underscore that this approach gives persistent attention to the integrand quantity as an accumulation rate with respect to a desired quantity (and unit), and the differential and interval of accumulation are constructed in tandem so that the written integral emerges in terms of and with respect to θ. Each is *expressed in terms of* and *with respect to* θ, and their symbolization co-emerge through determining the angle measure interval, determining the integrand quantity as an accumulation rate expressed in terms of and with respect to the angle measure in radians, and using a differential that is expressed in terms of and with respect to the angle measure in radians (i.e., the differential representing an accrual of radian angle measure). All expressions and implied arithmetic operations emerge through a quantitative understanding of the situation, its measures, and its unit magnitudes. Specific to the scalar $12/\pi$, it emerges in response to desiring the accumulation rate with respect to θ and identifying that a unit radian interval of accrual is $12/\pi$ times as large as the interval of accrual that occurs for a unit hour, as opposed to being a numerical value where its appropriateness is retrospectively justified using arithmetic operations and unit-tracking. The persistent quantitative attention taken in our approach makes it a quantitively organic answer to the question: “What if we wanted to track the total energy in terms of the *angle* the sun makes with the horizon (in radians), rather than the time on the clock?” (Jones & Fonbuena, 2024, p. 915)

5.2 Making Integration by Substitution Salient

How might our alternative approach be represented using the traditional integration by substitution method? Because we desire to express the accumulation rate with respect to angle measure rather than time, we relate their measures. We have $\theta = \frac{\pi}{12}t$. Because we desire to express the accumulation rate with respect to radians rather than hours, we also need the rate relationship between the two (i.e., an accrual relationship), which is $d\theta = \frac{\pi}{12}dt$. From these, we know the following quantitative properties:

1. A radian angle measure is $\pi/12$ times as large as its corresponding time measure in hours. An elapsed time in hours is $12/\pi$ times as large as its corresponding radian angle measure.
2. For an arbitrary accrual interval measure of elapsed hours, the accrual interval measure in radians is $\pi/12$ times as large. For an arbitrary accrual interval measure in radians, the accrual interval measure in elapsed hours is $12/\pi$ times as large.
3. Property 2 simultaneously means the accumulation interval for a unit radian is $12/\pi$ times as large as that for a unit hour. The accumulation interval for a unit hour is $\pi/12$ times as large as that for a unit radian.

We thus know that if energy accumulates at $1500\sin\left(\frac{\pi}{12}t\right)$ kJ/hr, it also accumulates at $\left(\frac{12}{\pi}\right)1500\sin(\theta)$ kJ/rad. We know the accumulated amount must be equivalent regardless of being expressed in terms of or with respect to time or angle measure, so we have

$$\int_0^6 1500\sin\left(\frac{\pi}{12}t\right)dt = \int_0^{\pi/2}\left(\frac{12}{\pi}\right)1500\sin(\theta)d\theta.$$

This equation emerges from our quantitative understanding of the situation, rather than as a multiple-part "conversion" or "substitution" using the previous expression. It foregrounds trying to represent the same accumulated energy with respect to and in terms of θ rather than t.

5.3 What About Riemann Sums?

We were confident in the quantitative structure of our alternative approach when fleshing out our initial pass at this conceptual analysis. We were, however, skeptical as to the extent it would relate to a popular method to building integration: Riemann Sums. We return to the energy context to compare the two conceptual analyses approaches with respect to Riemann Sums. Energy accumulates at a rate of $1500\sin(\pi t/12)$ kJ/hr, and we are interested in how much energy accumulates from 0 hours to 6 hours. For simplicity's sake, we use a (left) Riemann Sums approach with 3 intervals, obtaining

$$E \approx 1500\sin(0)\cdot 2 + 1500\sin\left(\frac{2\pi}{12}\right)\cdot 2 + 1500\sin\left(\frac{4\pi}{12}\right)\cdot 2.$$

Generalizing to a Riemann Sum for n intervals, we have

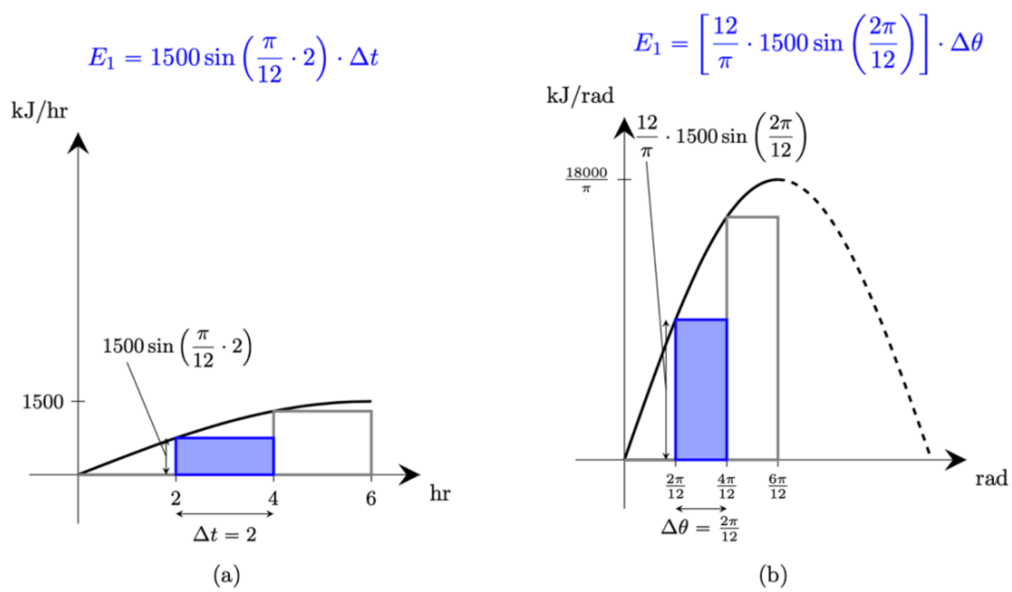
$$E \approx \sum_{i=0}^{n-1} 1500\sin\left(\frac{\pi}{12}t_i\right)\cdot\Delta t$$

with

$$t_i = 0 + i\Delta t \text{ and } \Delta t = \frac{6-0}{n}.$$

The Riemann Sums approach is an expression of accumulation in terms of the addition of accruals, with each accrual determined by the multiplicative relationship between an accumulation rate and accrual interval.

Figure 8. (a) Graphically representing the original Riemann Sum for $n = 3$, $i = 2$ and (b) the Riemann Sum in terms of and with respect to θ for $n = 3$, $i = 2$.



The focus of our approach is primarily on the accumulation rate. We have $\theta_i = \frac{\pi}{12}t_i$ and $\Delta\theta = \frac{\pi}{12}\Delta t$. Still desiring n intervals, we have $\theta_i = 0 + i\Delta\theta$ and $\Delta\theta = \frac{\pi/2}{n}$. Because a unit radian accrual interval is 12/π times as large as the accrual interval corresponding to a unit hour, we know energy accumulates at a rate of $\left(\frac{12}{\pi}\right)1500\sin\left(\frac{\pi}{12}t_i\right)$ kJ/rad. We thus express our Riemann Sum as $E = \sum_{i=0}^{n-1}\frac{12}{\pi}\cdot 1500\sin(\theta_i)\cdot\Delta\theta$. Mirroring our integration by substitution approach, the accumulation rate is expressed in kJ/rad (i.e., with respect to θ) and evaluated in terms of θ. The accumulation rate is then applied to an accrual interval of angle measure, $\Delta\theta$. Whereas the initial Riemann Sum expression relies on accrual intervals expressed in time, our resulting expression relies on accrual intervals expressed in angle measure. No regrouping of terms or symbolic moves are needed. Graphically, this process is represented beginning with Figure 8a and then reconstructing the energy accumulation as displayed in Figure 8b. Because the accumulation rate is expressed in terms of and with respect to θ, the Riemann Sum that parallels our conceptual analysis appropriately represents accumulation in the graphical approach. The two contained areas represent the same accumulated energy because, although the graphs involve different unit magnitudes, the graphs represent the same quantities with commensurate unit magnitudes within each graphical representation.[8]

[8] We have avoided referencing area to this point because Riemann Sums and integration are about accumulating quantities that just "happen" to be represented by area under a curve if the conditions are correct (e.g., using an orthogonal Cartesian coordinate system with appropriate units for all quantities). An accumulation rate that is with respect to time but expressed in terms of angle measure is not such a condition;

6. Comparing the Two Approaches

Our approach and the Jones and Fonbuena (2024) approach each capitalize on aspects of quantitative reasoning in detailing integration by substitution. For the sake of comparison, we draw attention to three notable ways in which our conceptual analysis differs from theirs. First, Jones and Fonbuena (2024) focused primarily on quantities' measures (or values) and arithmetic operations in the form of numerical conversions, whereas we persistently focus on magnitudes and measures. As an example, our approach involves multiplicatively comparing the accrual interval unit magnitudes for the purpose of determining the accumulation rate with respect to each unit magnitude. As another example, when Jones and Fonbuena (2024) spoke to "size," they were primarily referencing numerical measures; they appropriately reasoned that the number 2 (a time measure) is greater than the number 2π/12 (an angle measure). In our approach, we focus on "size" in terms of a magnitude, which allows for a span of angle measure to be considered the same "size" as a span of time in that they can each correspond to the same interval of sun movement and, hence, energy accumulation. This focus on magnitude was important in our approach because it made it possible to focus on the integrand quantity as an accumulation rate that can be expressed in terms of and with respect to different quantities for the same accumulation interval size.

A second, albeit related to the above, difference between the approaches is that make explicit two potential components to pursuing the question, "What if we wanted to track the total energy in terms of the *angle* the sun makes with the horizon (in radians), rather than the time on the clock?"[9] (Jones & Fonbuena, 2024, p. 915). Firstly, we can desire to *express* the accumulation *in terms of* θ, as opposed to t. Secondly, we can desire to *determine* the accumulated energy using the energy accumulation rate *with respect to* θ, as opposed to t. The approach by Jones and Fonbuena (2024) approached the question primarily using the first component, and in Fonbuena and Jones (2024) they referred to this as a conversion to "track the target quantity in terms of a new input quantity (e.g., θ), rather than the original input (e.g., t)" (p. 27). In our approach, we also give attention to the second component by incorporating Thompson and colleagues' (2014) ideas of quantity and unit-measure pairs.

An implication of this difference is in the emergence of the integral that results from integration by substitution. Each approach produces a correct expression. In the case of Jones and Fonbuena (2024), they followed a substitution and regrouping method that involves formulas between measures and manipulating symbolic forms. In our case, we follow an approach that is consistent with Carlson, O'Bryan, and Rocha's (2022) *emergent symbolization* or *emergent symbol meaning*. Carlson et al. (2022) described emergent symbolization as a set of beliefs and expectations for students that involves their productively understanding mathematical representations as capturing mathematical reasoning in the form of quantities and their relationships. Reflecting Stevens's (2019) extensive and detailed look into pre-service teachers' formula use, emergent symbolization draws attention to critically analyzing the extent formulas represent dynamic, generative, and flexible reasoning about relationships between quantities. As

the area under the curve is not numerically equivalent to the quantity's accumulation. See Izsák (2025) for an approach compatible with ours that alternatively foregrounds Riemann Sums and areas with attention to multiplication as coordinated measurement.

[9] In our closing discussion, we allude to the importance of creating an intellectual need for asking and pursuing this question.

Stevens (2019) argued, students' ability to productively produce and use formulas and variables is influenced by the extent that the coordination of quantities and their relationships persistently drives such activity. Echoing Izsák's (2025) recent analysis, we view our approach as providing an image of integration by substitution such that symbolization originates from quantitative reasoning and thus represents not only arithmetic operations between values, but also quantitative structures. Such emergent reasoning is critical to perceiving mathematical properties as remaining invariant across different values, contexts, and representations (Moore et al., 2022; Moore et al., 2024).

As a third point of difference, we do not rely on dimensional analysis or unit-tracking in our conceptual analysis. This was an intentional choice on our part due to our past work in the area (Moore et al., 2016), as well as us having been influenced by other researchers who have argued that dimensional analysis inhibits or masks critical aspects of quantitative reasoning (Reed, 2006; Thompson, 1994a). Despite our intentional avoidance of dimensional analysis or unit-tracking for the purpose of relying on quantitative reasoning and structures, the reader might use such techniques to further convince themselves of the expression structures in our approach.

7. Summary and Closing

We admit that our alternative approach rests on a quantitative sophistication a reader might deem overly complex or cumbersome, particularly in cases in which the reader is not overly familiar with quantitative and covariational reasoning. We do not agree that the quantitative complexity of our approach is unnecessarily complex or cumbersome. The growing importance of individuals' quantitative reasoning—whether those individuals are students, teachers, or public members (Gantt et al., 2023; Karagöz Akar et al., 2022; Yoon et al., 2021)—suggests such reasoning is a critical way to address what Thompson (2013) termed an "absence of meaning" from mathematics curriculum and instruction. It is through ignoring or leaving implicit the quantitative complexity of mathematics that it gets diluted to a collection of learned facts, procedures, and numerical calculations. It is through giving attention to and wrestling with the quantitative complexity of mathematics, which our approach and that of Jones and Fonbuena (2024) each accomplish to some degree, that students get to act as mathematicians through creating and understanding mathematical structures.

Relatedly, we do not intend this conceptual analysis to imply a rejection of the approach by Jones and Fonbuena (2024). We consider their approach viable, meaningful, and worth building upon. Furthermore, their approach inspired us to think deeply on this topic. We thus provide our conceptual analysis in order to contribute an approach that draws on alternative and more explicit aspects of quantitative reasoning, including those that we have personally found crucial in our work modeling students' and teachers' mathematical reasoning across a variety of major secondary concepts (e.g., Moore et al., 2016; Moore et al., 2022; Moore et al., 2024). The way to truly test and compare conceptual analyses is through the empirical work of building second-order models (Steffe & Thompson, 2000; Thompson, 2013). Doing so provides insights into the affordances and constraints of a conceptual analysis as related mathematical ideas are co-constructed by a researcher and their participants. In this respect, the conceptual analysis of Jones and Fonbuena (2024) is superior to the approach presented here, as they have conducted a teaching experiment to explore the viability of their approach and provide useful insights into student reasoning

(Fonbuena & Jones, 2024). We hope that in providing this alternative conceptual analysis, other researchers might pursue its potential by working with students in ways that make salient certain aspects of quantitative reasoning. Such work will shape our approach in response to models of student thinking, ultimately altering, discarding, or combining it with aspects of the approach by Jones and Fonbuena (2024).

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