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Expanding the Scope of Unit Transformation Graphs to Space and Measurement

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ABSTRACT

Building models of students' mathematics enables researchers to understand, appreciate, and support students' reasoning. Grounded in Piagetian epistemology, unit transformation graphs provide a way to model students' mathematics by explicating the mental actions that students use and coordinate in problem-solving. Prior research has focused on unit transformation graphs in numerical contexts, particularly fractions. The purpose of this theoretical article is to expand the scope of unit transformation graphs to model students' spatial comparison, enumeration, and measurement, by modeling students' coordinations of numerical and spatial mental actions. We illustrate this expansion with examples of one preservice elementary school teacher's solutions to tasks involving spatial quantities. We conclude with implications for both theory and practice.

Keywords: units coordination · spatial structuring · unit transformation graphs · length measurement · area measurement · volume measurement

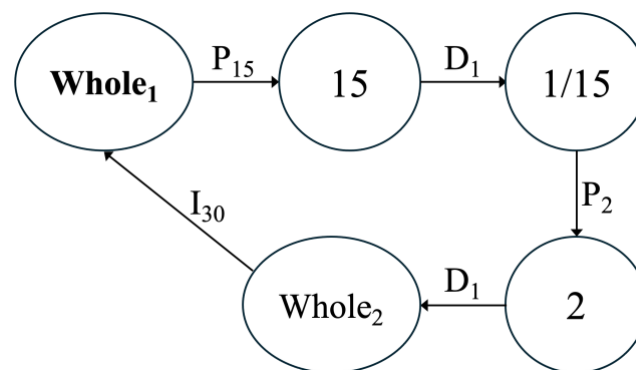
Within Piagetian epistemology, there is no distinction between mathematics and the psychology of mathematics (Noddings, 1990). From this point of view, all mathematical concepts, even as simple as 1, are psychological constructs—structures that fundamentally shape how we perceive, interpret, and operate within our experiential worlds. According to Piaget (1950), mental actions form the basis of all such structures, particularly within logic and mathematics.

Taking this epistemological stance seriously, Norton et al. (2023) introduced the notion of a *unit transformation graph* (UTG) as a tool for modeling students' mathematics as organizations of mental actions, with particular attention to fraction multiplication. UTGs are directed vertex-edge graphs where directed edges represent mental actions used to transform a unit (the source vertex) into another unit (the target vertex). To illustrate, consider the Cake Task.

Cake Task. Imagine a rectangular cake that is cut into 15 equal pieces. You decide to share your piece of cake fairly with one other person. How much of the whole cake would that person get? (adapted from Hackenberg & Tillema, 2009)

Figure 1 depicts a UTG of a hypothetical solution to the Cake Task (in the next section we elaborate on the notation used in this figure and other UTGs). The UTG illustrates how the problem-solver might take the cake as a whole, partition the cake into 15 equal-size pieces (P_{15}), and disembed 1 piece from the 15 pieces (D_1), forming $1/15$ of the whole. Then the student might partition that piece into 2 equal-size pieces (P_2) and disembed 1 of those parts (the piece to be given away). Next, the problem-solver might relate the size of the given-away piece to the whole cake by iterating this piece 30 times (I_{30}), forming a 1-to-30 relationship between the piece and the whole, and concluding that the piece measures $1/30$ of the cake.

Figure 1. UTG for the Cake Task, adapted from Norton et al. (2023).



UTGs like the one illustrated in Figure 1 do not inherently provide more information than paragraph-style narrative accounts of students' mathematics. Rather, we view UTGs as synergistic with narrative accounts. They are particularly useful additions for at least two reasons. First, UTGs provide a way to illustrate students' mathematics in terms of their coordination of mental actions. As such, they require researchers to carefully examine students' mental actions, highlighting aspects of students' reasoning that narrative accounts can contextualize and explain but otherwise might go unnoticed. Second, UTGs underscore our epistemological position that students' mathematics *is* mathematics. In other words,

because mathematics is indistinguishable from the psychology of mathematics, we can learn mathematics from students by attending to the mental actions students coordinate to solve problems.

Given the utility of UTGs for modeling students' mathematics, we argue that expanding the scope of UTGs to additional content areas is an important and worthwhile goal, contributing to constructivist researchers' available methods for building models. In particular, the purpose of this theoretical article is to expand the scope of UTGs beyond the realm of number and into the realm of spatial enumeration, comparison, and measurement. We contend that UTGs enable us to better represent students' mathematics by attending to both numerical and spatial actions.

In this theoretical paper, we begin by reviewing research related to units coordination and spatial structuring. We draw on that literature to argue for a unified perspective of mathematics, across number and space. With that unification in mind, we then illustrate our expansion of UTGs to spatial reasoning with an example. That example draws on data from a clinical interview conducted with a preservice elementary school teacher, Jake (pseudonym), and our analyses of Jake's reasoning in terms of his numerical and spatial mental actions.

Theoretical Framework and Related Literature

Our work is framed using Piaget's (1950) genetic epistemology. According to this perspective, mathematical concepts arise as coordinations of mental actions, where a *mental action* is a transformation that is composable and potentially reversible. Composability ensures the fruitfulness of mathematics, as students can compose mental actions to form new actions and solve ever-more-complicated tasks. For example, a student might compose mental actions of partitioning a whole into four pieces and iterating one of those pieces three times to produce $3/4$. Reversibility ensures the reliability of mathematics, as every action can be undone through another action, returning the student to the same initial state; though we note that students might engage with mental actions that are not yet reversible for them (hence, *potentially* reversible, from the researcher's perspective). For example, a student might partition a whole into four pieces without anticipating that they could reproduce the whole by iterating any of those pieces four times.

In this article, we refer to literature from two theoretical perspectives that are built upon Piaget's epistemology—units coordination and spatial structuring. For more detailed discussion of these two perspectives, see Antonides et al. (2026). In particular, Antonides et al. (2026) argued that although units coordination and spatial structuring have historically had limited interaction, the two perspectives share many of the same goals and constructs, afforded by their common epistemological grounding. Readers are also encouraged to see Hackenberg and Sevinc (2024) for discussion of units coordination, and Battista et al. (2018) for discussion of spatial structuring and related numerical reasoning.

The present work emerged from a larger, collaborative study (Antonides et al., 2026) that entailed bringing together two constructivist-based theoretical perspectives: units coordination (Hackenberg & Sevinc, 2024) and spatial structuring (Battista et al., 1998; Cullen et al., 2018). Spatial enumeration and measurement are closely linked to, yet qualitatively distinct from, the construction of numerical concepts and operations. The similarity of the two is demonstrated by the mental actions common to both domains

such as iterating, partitioning, disembedding, and distributing (defined in Table 1). However, there are also important differences, which Piaget recognized in his studies of children's development of length.

But unlike the unit of number, that of length is not the beginning stage but the final stage in the achievement of operational thinking. This is because the notion of metric [length] unit involves an arbitrary disintegration of a continuous whole. Hence, although the operations of measurement exactly parallel those involved in the child's construction of number, the elaboration of the former is far slower and unit iteration is, as it were, the capstone¹ to its construction. (Piaget et al., 1960, p. 149)

The eventual goal of our continuing work is to develop an integrated, unified theory of students' unit constructions and coordinations across number and space.

A Unified Perspective of Number and Space

The unity of number and space is evident throughout human experience. For the Pythagoreans of ancient Greece, the term *mathematics* meant the study of arithmetic and geometry (Burton, 2011); modern neuroscience indicates intricate relationships between spatial and numerical reasoning (Mix & Battista, 2018). Indeed, Winter et al. (2015) remarked, "Space, time, and number are fundamental to how we act within and reason about the world. These three experiential domains are systematically intertwined in behavior, language, and the brain" (p. 209). Within mathematics education, this unity is evidenced by common mental actions used to describe spatial and numerical reasoning, a topic to which we now turn.

Through extensive, longitudinal teaching experiments with children, Steffe leveraged Piagetian theory to carefully trace children's constructions of numerical concepts and operations, concepts that ultimately form the basis of children's fractional and multiplicative concepts (Steffe, 1992, 1994; Steffe et al., 1983; Steffe & Olive, 2010). This research program has been developed into a much-elaborated three-stage theory of students' numerical reasoning, called *units coordination* (Hackenberg & Sevinc, 2024; Hackenberg & Tillema, 2009; Norton & Boyce, 2015). Meanwhile, other mathematics education researchers leveraged Piagetian theory to study children's spatial reasoning. This work includes empirical investigations of children's mental processes for conceptualizing geometric shapes and measurement (Battista et al., 1998; Battista & Clements, 1996; Reynolds & Wheatley, 1996; Wheatley & Reynolds, 1996) resulting in the identification of progressions in children's conceptualizations and reasoning (e.g., Barrett et al., 2017; Battista, 2012; Sarama & Clements, 2009). In both research traditions, researchers model students' mathematics in terms their mental actions—thus, UTGs provide a powerful way to illustrate students' mathematics across both number and space.

Although there is much commonality between these two constructivist-based research communities, particularly in the theoretical constructs used to model students' mathematics, there has been little collaboration between them (as indicated by existing published research). We take the position that each community can learn from one another and, in so doing, contribute to our overall ability to model students' mathematics. On one hand, units coordination can benefit from research on spatial reasoning concerning ways that students construct and coordinate 2D and 3D units, and how students construct 2D and 3D space. On the other hand, spatial reasoning researchers can gain explanatory power by attending

¹ In Piaget et al. (1960), this term was translated as "coping stone." However, Piaget et al.'s (1948) actual phasing was *le faite d'une construction*, literally translated as "the ridge of a building" or the highest point of a roof, which we believe is more accurately captured by "capstone," the highest peak in a construction.

to the developmental stages and associated mental activities proposed within units coordination, especially regarding children's development of multiplicative reasoning and fractional concepts.

With an eye toward both numerical and spatial reasoning, we use the following definitions in our collaborative analyses of students' reasoning (see Antonides et al., 2026, for our analyses and discussion of our definitions). We define a *unit* to be a conceived entity taken as input for potential future action; examples of such actions include iterating, partitioning, translating, or rotating. Units are produced by *unitizing*, the mental act by which aspects of one's experience are isolated and treated as bounded entities through alternating pulses of focused and unfocused attention (von Glasersfeld & Richards, 1983). We distinguish qualitatively distinct types of units based on the mental actions used to construct and operate on them. *Numerical units* (or *units of one*) are entities that an individual conceptualizes as one and takes as input for numerical actions, such as iterating (e.g., to construct two) or partitioning (e.g., to construct one-half). *Spatial units* are entities that an individual takes as input for spatial actions such as rotating, reflecting, or translating; for instance, a rectangle in the plane could be constructed as a spatial unit. *Spatial/numerical units* are units that are both numerical and spatial, such as a geometric figure that an individual conceptualizes as one and can imagine operating on both numerically (e.g., by iterating it to create a plurality of such units) and spatially (e.g., by translating it along a linear dimension).

For example, a student might construct a square as a spatial/numerical unit if they iterate the square numerically (which might be indicated by counting, "One, two, three, ...") while creating copies of the square that are translated along a linear dimension, possibly forming a row or column of squares. *Composite units*, or *two-level units*, are the result of unitizing a collection of units (e.g., a row of squares conceptualized as a single entity). A *composite unit of composites units*, or *three-level unit*, is the result of unitizing a collection of composite units (e.g., a rectangular array mentally organized into rows with a fixed number of squares in each row).

Stages of Units Coordination

Mathematics education researchers taking a units coordination perspective (Hackenberg & Sevinc, 2024) have identified three qualitatively distinct stages of units-coordinating activity. Although contextual familiarity often impacts student reasoning, in general, the three stages of units coordination provide a model for explaining students' responses to mathematical problems across a variety of contexts. Students operating at Stage 1 can reliably take units of one as given, meaning they can assimilate (or interpret) problem situations in terms of units of one. Students at this stage can often anticipate a need to construct composite units, but composite units need to be constructed through explicit, effortful mental activity such as iterating and partitioning. Students operating at Stage 2 can take units of one and composite units as given; they can construct three-level units through effortful mental activity, but operating on all three levels of units at once is generally beyond the student's zone of potential construction (Steffe & D'Ambrosio, 1995) at this stage. Students operating at Stage 3 can take units of one, two-level units, and three-level units as given; they can move flexibly within three levels of units and can operate on all three levels of units at once.

Some students operating at Stage 2 demonstrate an ability to flexibly switch between different two-level unit structures among three levels of units but without operating on all three levels of units at once.

Hackenberg and Sevinc (2022) characterized such students as operating at *advanced* Stage 2. These students might be engaging in *daisy-chaining*, a form of reasoning where students “shift seamlessly from one two-level structure to the next, through the overlapping unit, but without coordinating all three units at once” (Norton et al., 2023, p. 252). As an example, students might rely on daisy-chaining to make unit conversions—for instance, 4 feet is equivalent to $4 \times 12 = 48$ in., which is equivalent to $48 \times 2.54 = 121.92$ cm, which is equivalent to $121.92 \div 100 = 1.2192$ m—chaining between two units of measure (feet to inches, inches to centimeters, centimeters to meters) at each step. In our UTGs, we refer to daisy-chaining to illustrate how advanced Stage 2 students might switch between two-level structures.

Spatial Structuring

Many constructivist-based mathematics education researchers focusing on students’ spatial enumerations and measurement have also focused on students’ unit constructions, but within spatial contexts (Barrett et al., 2017; Battista, 1999, 2004; Battista et al., 1998; Battista & Clements, 1996; Cullen et al., 2018; Piaget et al., 1960). Such researchers tend to focus on students’ *spatial structuring*, defined as “the mental act of constructing an organization or form for an object or set of objects,” often by “establishing units, establishing relationships between units (such as how they are placed in relation to each other), and recognizing that a subset of the objects, if repeated properly, can generate the whole set” (Battista & Clements, 1996, p. 282).

Much research points to the critical role played by spatial structuring in students’ ability to make sense of, and justify, spatial-unit enumeration strategies and formulas (e.g., Barrett et al., 2017; Battista, 2007; Battista et al., 2018). It is through spatial structuring that individuals conceptualize and mentally organize spatial objects in terms of their spatial components and parts, such as how squares are organized into rows or columns to form a 2D array (Battista et al., 1998), how cubes are organized into columns or layers to form a 3D cube building (Battista & Clements, 1996), or how tiles are organized within a tessellation of the plane (Antonides & Battista, 2023). Spatial structuring often begins through sensorimotor actions carried out on figurative material, but through reflection and abstraction, these actions can be internalized and organized to form mental operations (Antonides & Battista, 2022).

Spatial/Numerical Actions

Because research from both units coordination and spatial structuring theoretical perspectives has roots in Piagetian constructivism (Antonides et al., 2026), researchers from both communities often model students’ mathematics in terms of the same or similar mental actions. This commonality partially motivated our larger collaborative research effort between units coordination and spatial structuring perspectives. For instance, researchers from both communities often speak of iterating, partitioning, and distributing (Battista et al., 1998; Battista & Clements, 1996; Steffe, 1992). Although a full explication of these connections is beyond the scope of this article, we mention them to explain why many of the mental actions listed in Table 1 can be interpreted both numerically and spatially. These mental actions form the basis of the UTGs presented later in the article.

Table 1. Non-exhaustive list of numerical and spatial mental actions in spatial enumeration.

Mental Action	Notation in UTGs	Definition
Collapsing	Co	Reducing a higher-dimensional spatial object to a lower-dimensional object, such as collapsing or projecting a line segment to a point, or a plane to a line; the inverse mental action of sweeping a point to a line, or a line to a plane (Panorkou, 2020)
Decomposing	De	Breaking a spatial region into separate parts (Battista, 1999); also referred to as subdividing (Petitto, 1990; Piaget et al., 1960)
Disembedding	D_n	Removing n parts from a whole (composite unit) without mentally destroying the whole's integrity as a composite unit (Hackenberg, 2010)
Distributing	$T_{m:n}$	Inserting m units of a composite unit into each of n units of another composite unit (Steffe, 1992)
Iterating	I_n	Repeatedly instantiating a unit or part of a unit, creating n copies (Hackenberg, 2010); often composed with translating in spatial contexts (Barrett et al., 2017; Battista et al., 1998; Battista & Clements, 1996)
Partitioning	P_n	Breaking a numerical unit into n equal-size parts (Hackenberg, 2010); often composed with the spatial act of decomposing (Battista, 1999)
Rotating	Ro	Turning a spatial unit (Edwards, 1991)
Translating	Tr	Sliding a spatial unit from one location to another (Edwards, 1991)
Unitizing	U	Conceptualizing an item or collection of items as a single bounded entity (von Glasersfeld & Richards, 1983)

We provide this list of mental actions in a single table to underscore the commonality between the goals and constructs of the units coordination and spatial structuring perspectives.

Jake's Units Coordinating Activity

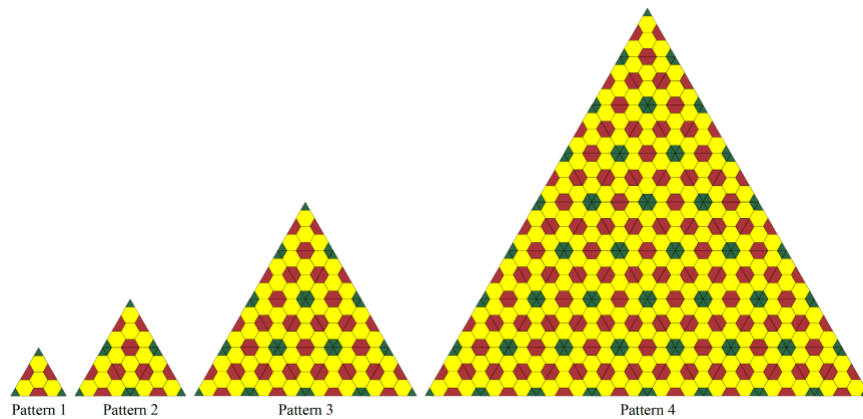
To illustrate our expansion of UTGs to space and measurement, we present an example from one 45-minute clinical interview with a pre-service elementary school teacher, Jake (pseudonym), who was a sophomore enrolled at a large public university in Southeastern United States. Jake was concurrently enrolled in a geometry course for future elementary school teachers, taught by the first author, at the university from which we recruited volunteers for a larger study (Antonides et al., 2026). Among the three students who participated in the larger study, we chose to report on Jake here because he was especially expressive in describing his thinking.

We inferred that Jake was operating at *advanced* Stage 2 of units coordination. We found evidence that Jake could assimilate tasks using two levels of units across all tasks (indicating at least Stage 2), but we found no evidence of Jake assimilating tasks using three levels of units. Evidence of Jake switching

between two-level structures (and thus evidence that he was operating at advanced Stage 2) was strongest in his response to the Pattern Tiles Task.

Pattern Tiles Task. The patterns below (see Figure 2) are made up of green triangles, red trapezoids, and yellow hexagons. (1) How many green triangles would it take to cover Pattern 2 with no gaps or overlaps? (2) How many green triangles would it take to cover the other patterns?

Figure 2. Images presented for the Pattern Tiles Task.



A detailed account of Jake’s reasoning on this task is provided by Antonides et al. (2026). In brief, Jake began by noting, “There’s 6 triangles in each hexagon. So, if you know that, and also same with the trapezoid one, it’s still a hexagon. It’s 2 trapezoids, though. So then that would be 6 triangles as well for each hexagon, as well as each trapezoid that’s in there. So—but some of them are cut off.” Jake then used the picture of Pattern 2 to iterate triangles as units of 1, trapezoids as units of 3, and hexagons as units of 6: “So, there’d be 1 [top triangle], 7 [hexagon below top triangle], 13 [trapezoid], 16 [trapezoid]. Um. 22 [hexagon],” and so on. He continued counting by 1s, 3s, and 6s until he reached 144 as the number of green triangles needed to cover Pattern 2.

We infer that Jake spatially decomposed one hexagon into a unit of six triangles, but he also knew that a hexagon represents a unit of two trapezoids: “There’s 6 triangles in each hexagon... It’s 2 trapezoids, though.” Jake thus established each shape as a representation of a different two-level unit structure—a hexagon as a unit of 6 triangles, a hexagon as a unit of 2 trapezoids, and a trapezoid as a unit of 3 triangles. His enumeration indicates that he could flexibly switch between these three distinct two-level unit structures, suggesting that he was operating at advanced Stage 2 of units coordination. In Part 2 of the Pattern Tiles Task, however, Jake did not assimilate with three levels of units, and in his subsequent composite-unit iterations Jake made frequent pauses and relied on visual cues from figurative material. We interpreted this as evidence of Jake not yet operating at Stage 3 of units coordination.

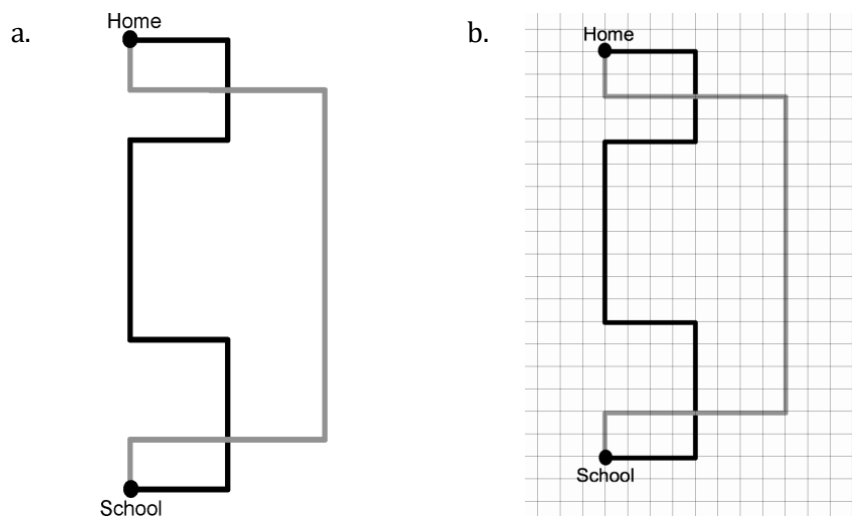
Methodology

Data Collection

Although the present article is theoretical in nature, we provide background information about how data from our larger study were collected and analyzed. Jake participated in one 45-minute clinical interview in which he was given four tasks. We used the Pattern Tiles Task to provide evidence that Jake operated at advanced Stage 2 of units coordination (see previous section). We use two additional tasks to illustrate the utility of UTGs across spatial contexts: a length comparison task (Sidewalk Task Part 1), a length-unit enumeration task (Sidewalk Task Part 2), and a volume-unit enumeration task (Packages Task). We chose these tasks for this report because they involve spatial enumeration at the two extremes: 1D and 3D.

Sidewalk Task. Which sidewalk from home to school is longer—the black one, the gray one, or are they the same length? (adapted from Battista, 2012)

Figure 3. Images presented for the Sidewalk Task.



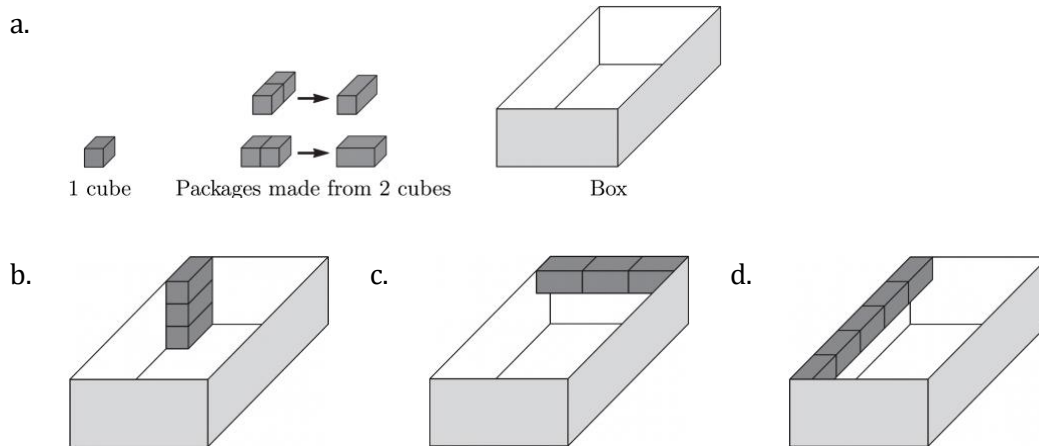
The Sidewalk Task consists of two parts: in Part 1, students are presented with an image that does not contain a grid (Figure 3a); in Part 2, they are presented with an otherwise identical image containing a grid (Figure 3b). Versions of the Sidewalk Task have been used extensively in prior research, both with elementary school children (Battista, 2003, 2012) and with practicing teachers (Battista, 2010).

The Packages Task has also been extensively used in prior research with children (Battista, 2004); in particular, Battista has used this task to assess whether students can apply and make sense of volume measurement formulas for arrays in situations involving non-cubic units.

Packages Task. Colin has some packages that each contain two identical cubes. He wants to know how many of these packages it takes to completely fill the rectangular box below (Figure 4a). He knows that 3 packages fit along the height of the box cubes (Figure 4b). He knows that

3 packages fit along the width of the box cubes (Figure 4c). He knows that 5 packages fit along the length of the box cubes (Figure 4d). How many packages would it take to fill the box? (Battista, 2004)

Figure 4. Images presented for the Packages Task.



Data Analysis

Data analysis entailed a multi-step process. Researchers split into two teams: the numerical team, which focused on units coordination and included the second author; and the spatial team, which focused on spatial reasoning and included the first author. Each team used existing research literature to generate a codebook consisting of particular mental actions, such as those in Table 1. Each researcher then individually coded data (e.g., video recordings, transcripts) using their team's codebook of mental actions. Then, researchers met within their teams to discuss interpretations of Jake's reasoning and resolve discrepancies in coding. Finally, researchers engaged in a four-part process toward generating models of Jake's mathematics that integrate both theoretical perspectives. A more complete account of our methodology and results is provided in Antonides et al. (2026); here, we draw on a subset of our findings to illustrate how we can model Jake's mathematics through UTGs.

Using UTG-construction methods of Norton et al. (2023), we use circles to indicate the units that students construct, and we use arrows to indicate mental actions that students use to transform those units. In the current paper, we extend those methods to include both numerical *and spatial* descriptors of the units that students construct, along with both numerical and spatial mental actions students use to transform them. We use rectangles to indicate two-level unit structures, but for ease of reading diagrams, we do not illustrate three-level units constructed in activity (though these are described in text). To aid the reader in interpreting the diagram, we indicate the beginning of Jake's solution using bold text.

Findings: Illustrating UTGs Across Spatial Contexts

In line with the purpose of this paper, and following the methods described above, we draw upon Jake's responses to two spatial comparison and enumeration tasks to generate UTGs that illustrate the spatial and numerical mental actions that constitute Jake's mathematics.

Illustrating Jake's Solutions to the Sidewalk Task

To solve Part 1 of the Sidewalk Task (see Figure 3a), Jake initially reasoned that the black sidewalk would be longer than the gray sidewalk because it contained more turns:

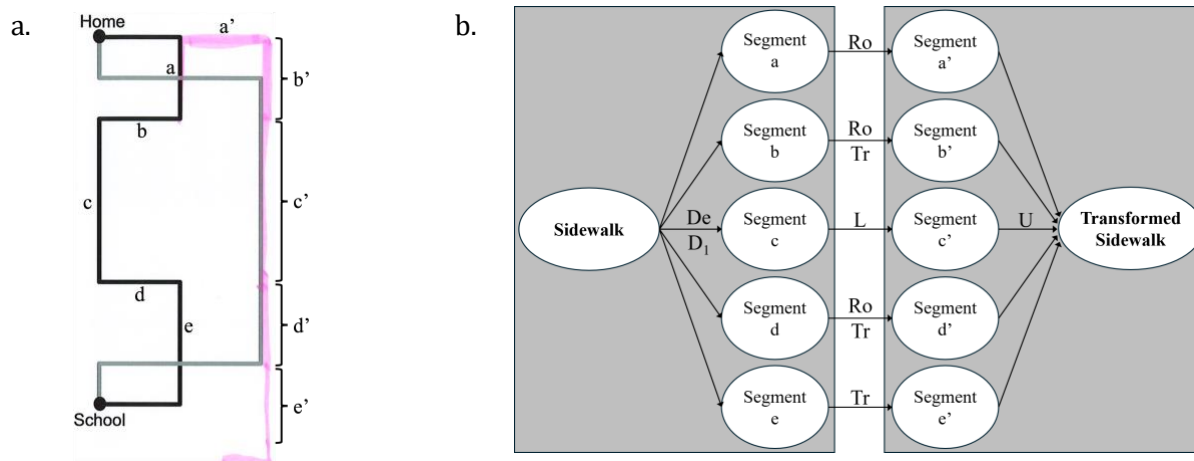
Jake: I feel like the black one would be longer, more, just because it has the switchbacks in it. So that you're turning a lot, where that [the gray sidewalk] you just turn once and go kind of around. ... If you like stretch this line out [black sidewalk] in the same shape, it would actually go wider than that one [gray sidewalk].

When the interviewer asked, "Is there any way that you could be sure, or think about the problem so that you could say definitively that the black one is longer than the gray one?", Jake took a different approach. He broke the sidewalk into separate parts and spatially acted on each part separately, making the black sidewalk into a form that he could more easily compare to the gray sidewalk.

Jake: So I'm kind of like breaking up that segment right here [top vertical black segment] and, like, turning it, and it would be to that edge [segments a and a' in Figure 5a]. ... And then that would leave you with this part [segment b], which is the same length. So it would essentially be right here [used highlighter to draw segment b'].

Just continued to mentally decompose the black sidewalk and transform it part by part, as represented by the highlighter markings in Figure 5a. As elaborated below and as illustrated in Figure 5b, we infer that Jake constructed the sidewalk as a spatial composite unit containing five length units; he then disembedded unit by unit to act on each unit, forming a new sidewalk as a composite unit of five length units.

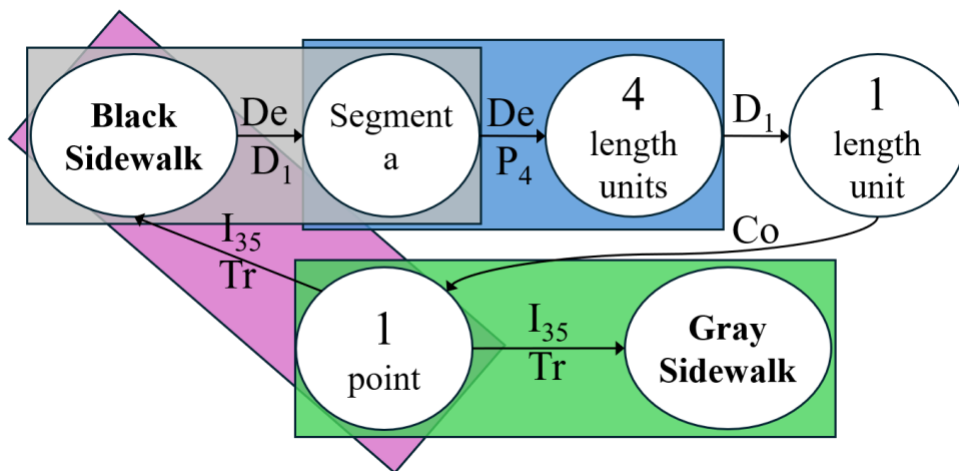
Figure 5. (a) Jake's part-by-part decomposition and transformation of the sidewalk and (b) UTG of Jake's solution to the Sidewalk Task Part 1.



We take Jake's suggestion of "stretch[ing] this line out [the black sidewalk]" as evidence that he treated the black sidewalk, in its entirety, as a spatial unit on which he could operate—in this case, by "stretching." We represent the original black sidewalk as the left-most circle in the UTG in Figure 5b. To stretch the black sidewalk, Jake took apart the black sidewalk and reassembled it in a different form that allowed him to compare its length to that of the gray sidewalk. To do so, we infer that Jake decomposed (De) the black sidewalk into five separate parts (segments labeled a-e), where these parts together make the whole sidewalk. This indicates that Jake constructed the sidewalk not just as a spatial unit, but as a spatial *composite* unit since the sidewalk consisted of a plurality of individual parts, each conceptualized as a spatial unit on which he could operate. This is indicated by the rectangle on the left-hand side of Figure 5b. He then disembedded each part one by one (D_1) and acted on it by translating (Tr) and/or rotating (Ro) it, evidenced by his verbal descriptions and highlighter markings (Figure 5a). Jake unitized (U) the collection of all spatially-transformed length units (labeled a'-e') into a single entity—a transformation of the black sidewalk—indicating that he treated this new sidewalk as a spatial composite unit. This is represented by the gray rectangle on the right-hand side of Figure 5b.

When presented with the Sidewalk Task Part 2 and the image in Figure 3b, Jake said that the grid "gives you kind of a key to go based off of." He focused on grid lines and, in particular, on "points" formed by grid-line intersections. For example, focusing on the top vertical segment of the black sidewalk (segment a in Figure 5a), Jake first said, "You can see obviously like that's four grids across." A moment later, Jake referred to the same segment as "4 points, essentially." He then counted points along the black and gray sidewalks, marking dots with a highlighter as he counted, concluding, "They're both 35 points." Jake thus used a point-counting strategy to compare the two sidewalks' lengths, concluding they are equal.

Figure 6. UTG of Jake's solution to the Sidewalk Task Part 2.



We model Jake's solution to Part 2 using the UTG shown in Figure 6. Jake took the black sidewalk as a spatial unit on which he could act. We infer that Jake decomposed the black sidewalk into separate segments (De), likely carrying over from Part 1, and established the sidewalk as a spatial composite unit (a unit of five length units). This is indicated by the gray rectangle in the UTG in Figure 6. Jake then disembedded segment a (using the labeling in Figure 5a) from the sidewalk (D_1) and partitioned it into

four equal-size length units (P_4): “that’s four grids across.” This is the first instance in which Jake presents evidence of giving numerical meaning to spatial units. From this, we infer that Jake conceptualized segment a as a *spatial/numerical* composite unit (represented by the blue rectangle in Figure 6), consisting of four length units, where each length unit is a spatial/numerical unit of one. Note that although the grids themselves *suggest* a partitioning of the sidewalk, Jake must still mentally construct this partitioning himself. Indeed, instead of “four grids” (by which we infer he meant four length units), Jake could have imagined a partitioning of segment a into five points or four squares—both common strategies exhibited by children (Battista, 2012).

Jake then transitioned from using length units to using points. As Jake was pointing to each sidewalk, he made several sweeping gestures with his finger. We take Jake’s sweeping gestures along each path, coupled with his transition from focusing on length units to points, as evidence that he imagined collapsing (Co) a length unit to its endpoint—likely by disembedding (D_1) a length unit first as material on which to operate. Eventually, Jake’s attention was entirely on points, treating them as spatial/numerical units and as proxies for length units themselves. He iterated 35 points (I_{35}) along both sidewalks, establishing them as spatial/numerical composite units (each a unit of 35 point-units, represented by the green and purple rectangles in Figure 6). Noticing that both contain the same number of points, Jake concluded, “They’re actually the same.” Additionally, the UTG in Figure 6 suggests that Jake’s strategy involved daisy-chaining. As indicated by the gray and blue rectangles, Jake shifted between two different two-level unit structures, chained together by a common unit: between the sidewalk and length segments (including segment a), then between segment a and four length sub-units. This gives evidence toward our hypothesis that students operating at advanced Stage 2 of units coordination may engage in daisy-chaining when operating with multiple levels of units.

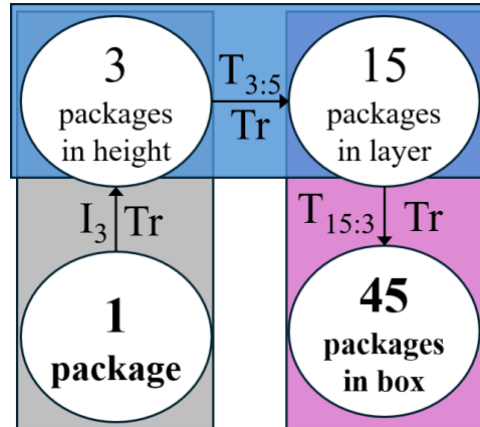
Illustrating Jake’s Solutions to the Packages Task

When presented with the Packages Task, Jake responded as follows: “So, he can fit three rows of 15. So 15, because I know that, because there’s three tall, but... Actually, those are turned in different ways. Nevermind. ... So yeah, I was kind of thinking the width is three and length is five.” Jake seemed to initially multiply the number of packages along one linear dimension by the number of packages along another linear dimension, multiplied by the number of “rows” (vertical layers). In other words, Jake seemed to be thinking about (3 along height) times (5 along length), finding 15 in one row covering the left-hand wall (stacked vertically), which he then planned to multiply by three to find “three rows of 15.” However, he realized this approach would not work because the packages along the width were “turned in different ways.”

Although Jake quickly abandoned his initial approach—an approach that many students take when presented with the Packages Task (Battista, 2004)—we take the opportunity to illustrate this approach with a UTG, displayed in Figure 7. We infer that Jake established a package as a spatial/numerical unit of one and constructed the height of the box (“there’s 3 tall”) as a spatial/numerical composite unit of three packages, indicated by the top rectangle in Figure 7. With the images provided, and with the numbers stated in the task statement, there is little evidence to suggest that Jake *explicitly* iterated and translated three packages to form the height. However, given our previous work with Jake, it seems highly likely that

he could have taken this iteration and translation for granted. Similarly, we infer that Jake established the length of the box as a spatial/numerical composite unit of five packages, and the width of the box as a spatial/numerical composite unit of three packages, both of which he could have constructed by iterating (I) and translating (Tr).

Figure 7. UTG of Jake's initial response to the Packages Task.



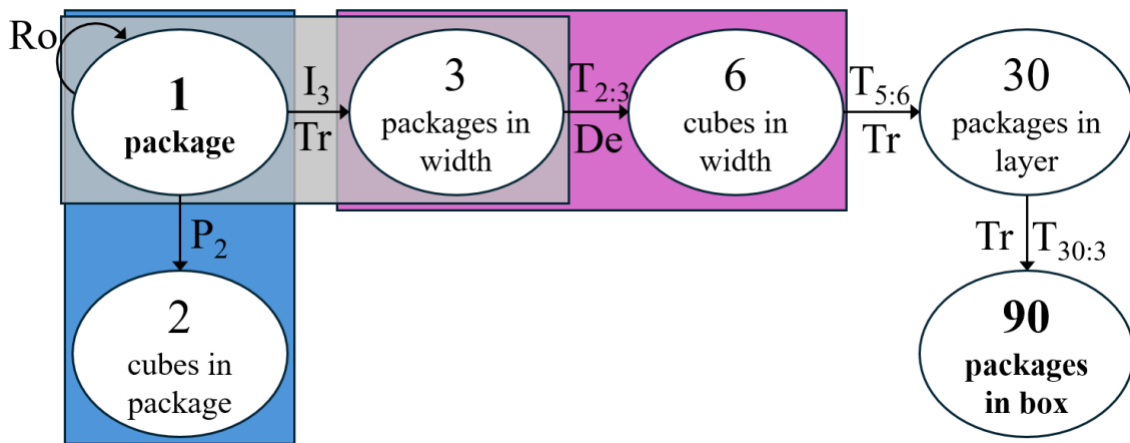
Then, we infer that Jake distributed the packages along the height over the packages along the box's length ($T_{3:5}$), creating a unit of 15 packages. Spatially, we infer that Jake's distributing action was coordinated with a translating action (L), iteratively translating three stacked packages indexed by the packages along the box's length. This action would give spatial meaning to the numerical value 15, and it seems plausible that Jake constructed 15 as a spatial/numerical composite unit (rather than just a numerical unit) because he seemed to imagine distributing this unit of 15 over the unit of three packages along the box's width, creating an initial prediction that 45 packages can fit inside the box. It was in carrying out this mental distribution that we infer Jake experienced a brief perturbation, where he realized that the three packages along the width were "turned in different ways." Indeed, this is an instance where Jake's spatial mental action of rotating, and his construction of 15 packages as not just a numerical composite but a *spatial*/numerical composite, were critical to him avoiding an incorrect solution.

Revising his response, Jake focused on the three packages along the width of the box: "If you look at this, when you have it turned, this would actually be six of these rows. Because the packages are made up of two cubes." He then reasoned: "So, if you can fit five this way [along the length]. [long pause] Yeah, there'll be six rows of five. So there would be 30 total. On the top." He then multiplied 30 by three, concluding 90 packages could fit in the box.

We represent Jake's revised strategy using the UTG in Figure 8. Aided by figurative material (Figure 4a), we infer that Jake partitioned (P_2) each package spatially and numerically into "two cubes," constituting each package as a spatial/numerical composite unit (indicated by the blue rectangle in Figure 8). He also constructed the width as a spatial/numerical composite unit of three packages (gray rectangle in Figure 8), which we infer he did by mentally iterating (I_3) and translating (Tr) one package along the width. With this, Jake realized that because three *horizontal* packages fit along the width ("when you have it turned"), he could fit six rows (oriented front-to-back) along the width. From Jake's determination that

six rows of packages could fit along the width, we infer that Jake spatially decomposed (De) each package into two cubes and distributed a unit of two units (two cubes in one package) over each of the three packages along the width ($T_{2:3}$), forming a spatial/numerical composite unit of six cubes (purple rectangle in Figure 8). Taking the packages along the length as a spatial/numerical composite unit of five packages, he distributed this unit of five packages into each of the units in the composite of six cubes ($T_{5:6}$) while mentally translating along the width (L), effectively using cubes as indicators for the placement of composite units of five packages. In doing so, Jake formed a 30-package layer as a spatial/numerical composite unit. He then distributed (and translated) this layer over the packages along the height of the box ($T_{30:3}$, L), concluding that the box can contain 90 packages.

Figure 8. UTG of Jake’s revised response to the Packages Task.



Discussion and Conclusions

Our work affirms the unity of mathematics found in history, neuroscience, and human behavior (Burton, 2011; Mix & Battista, 2018; Winter et al., 2015). We argue that this fundamental unity of mathematics depends on relationships between numerical units and spatial units, and we have offered definitions of numerical and spatial units to emphasize such commonalities. Furthermore, in promoting a Piagetian perspective, which makes no distinction between mathematics and the psychology of mathematics (Piaget, 1950), we foreground the mental actions that students use to construct and transform units.

Our framing of numerical and spatial units as objects that humans can act upon through further mental action enabled us to illustrate students’ mathematics using UTGs in a more unified way. As Norton et al. (2023) argued, “UTGs explicitly recognize students’ mental actions as the basis for mathematics itself” (p. 254). They take seriously the ontological commitments of Piaget’s genetic epistemology, while bringing to the fore students’ mathematics by carefully tracing their mental actions and coordinations thereof. With a common basis for spatial and numerical units, we were able to model students’ spatial enumeration and measurement by tracing their unit constructions and transformations, accounting for both numerical and spatial actions.

Prior accounts of units coordination have tended to focus on numerical actions alone (Hackenberg & Sevinc, 2024; Steffe, 1992), without explicit attention to spatial transformations, such as rotating and translating. These two mental actions were critical for Jake to solve the two spatial comparison and enumeration tasks that we shared. Jake began making progress on the Sidewalk Task by making correspondences between segments within a composite unit, then by rotating and translating them. For the Packages Task, his rotation of the package led him to transform one spatial/numerical unit into another (packages into cubes), then compose numerical and spatial mental actions together (e.g., distributing composed with translating), which enabled him to correctly enumerate the number of packages in the box. With a unified definition of unit, and by attending to Jake's numerical and spatial mental actions, we were able to provide a more complete account of Jake's mathematics. Furthermore, we were able to represent that mathematics within UTGs that specified each of the mental actions he used to solve the tasks.

Prior research has documented difficulties that students experience with respect to Part 2 of the Sidewalk Task (Battista, 2003, 2012) and the Packages Task (Battista, 2004, 2012). Students' difficulties have been attributed, in part, to students not sufficiently decomposing spatial units into separate components (e.g., grid-squares into length units), and not spatially structuring spatial units to form spatial composites that can be operated on spatially in ways that are linked to their arithmetic operations. We argue that units coordination supplements these explanatory accounts. Students who are not yet operating at Stage 3 of units coordination need to build up three-level unit structures in mental activity, competing for cognitive resources that might otherwise be used in other aspects of problem solving. In the Packages Task, Jake needed to actively build up the box as a unit of 3 units (layers) of 30 units (packages). Moreover, he actively built 1 layer as a unit of 6 units (rows) of 5 units (packages). Tracking the two- and three-level units that students construct, and how they construct them, gives important insight into students' spatial enumerations and reasoning.

UTGs also allowed us to illustrate Jake's use of daisy-chaining (Norton et al., 2023) to coordinate two two-level unit structures with an overlapping unit, both on the Sidewalk Task (Figure 6) and on his two responses to the Packages Task (Figure 7 and Figure 8). We suspect that Jake's use of figurative material, particularly in representing spatial-numerical structures, facilitated this way of operating. For example, in his initial response to the Packages Task, Jake conceived of the three-part stack as a unit constructed from three iterations of a package unit; he then used this three-part structure (represented as a stack) to distribute three units into each of five stacks across the length of the box.

Recall that, at the time of our study, Jake was concurrently enrolled in a geometry course for elementary school teachers. Elementary teacher education programs commonly offer such a course, paired with a complementary course like numbers and operations (Masingila et al., 2012). These courses present mathematics as a bifurcated field: number and geometry, separate from one another. Although the potential for our definitions and expansion of UTGs to impact teacher education needs further research, we note that our expansion of UTGs illustrates how spatial and numerical mental actions can be used together in problem solving—a potentially instructive illustration for preservice teachers. Moreover, our expansion of UTGs reflects the unity of number and space, along with the theoretical cohesion between units coordination and spatial structuring perspectives (Antonides et al., 2026).

Relatedly, UTGs foreground mental actions and how they are coordinated, with no mention of rules or procedures. With this emphasis, UTGs could be used in the classroom to model children's problem-solving or preservice teachers' own problem-solving, and to illustrate different solution paths for the same problem. As illustrated in the UTGs we have shared, these solution paths can be quite complex and sensitive to the way tasks are posed. For instance, consider Parts 1 and 2 of the Sidewalk task, which differed by the presence of a grid. The UTGs we present for Jake's solutions to these two parts of the task (see Figures 5 and 6) differ considerably in terms of the spatial and numerical actions that Jake coordinated.

Above all, UTGs underscore the fundamental role played by mental actions in constructing mathematics, which could support two fundamental shifts in teachers' views of mathematics. The first is the aforementioned shift away from viewing mathematics a collection of distinct, disjoint "silos," toward a more unified vision of mathematics. The second is a shift away from viewing mathematics as a body of knowledge that exists separate from students, toward a view of mathematics as a product of students' own activity. Prior research indicates that many pre-service teachers maintain the former views (e.g., Polly et al., 2013). By presenting mathematics as a body of knowledge unified by the coordination of students' own mental actions, UTGs might promote the latter views.

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References

- Antonides, J., & Battista, M. T. (2022). Spatial-temporal-enactive structuring in combinatorial enumeration. *ZDM Mathematics Education*, 54(4), 795–807. <https://doi.org/10.1007/s11858-022-01403-0>
- Antonides, J., & Battista, M. T. (2023). Students' enumerations of tiles within a rotation-based tiling of the plane: Juxtaposing units coordination and spatial structuring perspectives. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 385–393). University of Nebraska Omaha.
- Antonides, J., Norton, A., Battista, M. T., & Zwanch, K. (2026). Spatial structuring and units coordination: Building and illustrating a coordinated theoretical framework. *Cognition and Instruction*, 44(3), 271–314. <https://doi.org/10.1080/07370008.2025.2571509>
- Barrett, J. E., Clements, D. H., & Sarama, J. (2017). *Children's measurement: A longitudinal study of children's knowledge and learning of length, area, and volume*. National Council of Teachers of Mathematics.
- Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education*, 30(4), 417–448. <https://doi.org/10.2307/749708>

- Battista, M. T. (2003). Levels of sophistication in elementary students' reasoning about length. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th conference of PME-NA* (Vol. 2, pp. 73–80). University of Hawai'i.
- Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning*, 6(2), 185–204. https://doi.org/10.1207/s15327833mtl0602_6
- Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Information Age Publishing.
- Battista, M. T. (2010). *Teachers' conceptualizations of students' mathematical thinking and their consequent design of instruction: A cognitive perspective*. Annual meeting of the AERA, Denver, CO.
- Battista, M. T. (2012). *Cognition-based assessment and teaching of geometric measurement: Building on students' reasoning*. Heinemann.
- Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education*, 27(3), 258–292. <https://doi.org/10.2307/749365>
- Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education*, 29(5), 503–532. <https://doi.org/10.5951/jresmetheduc.29.5.0503>
- Battista, M. T., Frazee, L. M., & Winer, M. L. (2018). Analyzing the relation between spatial and geometric reasoning for elementary and middle school students. In K. S. Mix & M. T. Battista (Eds.), *Visualizing mathematics: The role of spatial reasoning in mathematical thought* (pp. 195–228). Springer. https://doi.org/10.1007/978-3-319-98767-5_10
- Burton, D. M. (2011). *The history of mathematics: An introduction* (7th edition). McGraw-Hill.
- Cullen, A. L., Eames, C. L., Cullen, C. J., Barrett, J. E., Sarama, J., Clements, D. H., & Van Dine, D. W. (2018). Effects of three interventions on children's spatial structuring and coordination of area units. *Journal for Research in Mathematics Education*, 49(5), 533–574. <https://doi.org/10.5951/jresmetheduc.49.5.0533>
- Edwards, L. D. (1991). Children's learning in a computer microworld for transformation geometry. *Journal for Research in Mathematics Education*, 22(2), 122–137. <https://doi.org/10.2307/749589>
- Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction*, 28(4), 383–432. <https://doi.org/10.1080/07370008.2010.511565>
- Hackenberg, A. J., & Sevinc, S. (2022). A boundary of the second multiplicative concept: The case of Milo. *Educational Studies in Mathematics*, 109(1), 177–193. <https://doi.org/10.1007/s10649-021-10083-8>
- Hackenberg, A. J., & Sevinc, S. (2024). Students' units coordinations. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), *Piaget's genetic epistemology for mathematics education research* (pp. 371–411). Springer.

- Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *Journal of Mathematical Behavior*, 28(1), 1–18. <https://doi.org/10.1016/j.jmathb.2009.04.004>
- Masingila, J. O., Olanoff, D. E., & Kwaka, D. K. (2012). Who teaches mathematics content courses for prospective elementary teachers in the United States? Results of a national survey. *Journal of Mathematics Teacher Education*, 15(5), 347–358. <https://doi.org/10.1007/s10857-012-9215-2>
- Mix, K. S., & Battista, M. T. (Eds.). (2018). *Visualizing mathematics: The role of spatial reasoning in mathematical thought*. Springer. <https://doi.org/10.1007/978-3-319-98767-5>
- Noddings, N. (1990). Constructivism in mathematics education. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 7–18). National Council of Teachers of Mathematics.
- Norton, A., & Boyce, S. (2015). Provoking the construction of a structure for coordinating $n+1$ levels of units. *The Journal of Mathematical Behavior*, 40, 211–232. <https://doi.org/10.1016/j.jmathb.2015.10.006>
- Norton, A., Ulrich, C., & Kerrigan, S. (2023). Unit transformation graphs: Modeling students' mathematics in meeting the cognitive demands of fractions multiplication tasks. *Journal for Research in Mathematics Education*, 54(4), 240–259. <https://doi.org/10.5951/jresmetheduc-2021-0031>
- Panorkou, N. (2020). Dynamic measurement reasoning for area and volume. *For the Learning of Mathematics*, 40(3), 9–13.
- Petitto, A. L. (1990). Development of numberline and measurement concepts. *Cognition and Instruction*, 7(1), 55–78. https://doi.org/10.1207/s1532690xci0701_3
- Piaget, J. (1950). *Introduction à l'épistémologie génétique: La pensée mathématique* [Introduction to genetic epistemology: Mathematical thinking]. Presses Universitaires de France.
- Piaget, J., Inhelder, B., & Szeminska, A. (1960). *La géométrie spontanée de l'enfant* [The child's conception of geometry]. (E. A. Lunzer, Trans.) Presses Universitaires de France. [Original work published 1948]
- Polly, D., McGee, J. R., Wang, C., Lambert, R. G., Pugalee, D. K., & Johnson, S. (2013). The association between teachers' beliefs, enacted practices, and student learning in mathematics. *The Mathematics Educator*, 22(2), 11–30.
- Reynolds, A., & Wheatley, G. H. (1996). Elementary students' construction and coordination of units in an area setting. *Journal for Research in Mathematics Education*, 27(5), 564–581. <https://doi.org/10.2307/749848>
- Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), 259–309. [https://doi.org/10.1016/1041-6080\(92\)90005-Y](https://doi.org/10.1016/1041-6080(92)90005-Y)
- Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 1–41). State University of New York Press.

- Steffe, L. P., & D'Ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. *Journal for Research in Mathematics Education*, 26(2), 146–159. <https://doi.org/10.5951/jresmetheduc.26.2.0146>
- Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. Springer.
- Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. Praeger.
- von Glasersfeld, E., & Richards, J. (1983). The creation of units as a prerequisite for number: A philosophical review. In L. P. Steffe, E. von Glasersfeld, J. Richards, & P. Cobb, *Children's counting types: Philosophy, theory, and application* (pp. 1–20). Praeger.
- Wheatley, G. H., & Reynolds, A. (1996). The construction of abstract units in geometric and numeric settings. *Educational Studies in Mathematics*, 30(1), 67–83. <https://doi.org/10.1007/BF00163753>
- Winter, B., Marghetis, T., & Matlock, T. (2015). Of magnitudes and metaphors: Explaining cognitive interactions between space, time, and number. *Cortex*, 64, 209–224. <https://doi.org/10.1016/j.cortex.2014.10.015>